LECTURE 11.8: POWER SERIES

A **power series** is a series of the form

called coefficients of the power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where $\{c_0, c_1, c_2, \dots\}$ is a sequence of numbers

The number a is called the center of the series.

Examples:

	center	coefficients
1. $\sum_{n=0}^{\infty} 2(x-1)^n = 2 + 2(x-1) + 2(x-1)^2 + \dots$	1	$c_n = 2$
2. $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n^2+2} = \frac{1}{2} + \frac{x-4}{3} + \frac{(x-4)^2}{11} + \dots$	4	$c_n = \frac{1}{n^2+2}$
3. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{5^n} = 1 + \frac{x+2}{5} + \frac{(x+2)^2}{25} + \dots$	-2	$c_n = \frac{1}{5^n}$

CONVERGENCE EXAMPLE

Example: Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = 1 + \frac{x}{3} + \frac{x^2}{9} + \dots$$

where plugging in different values of x gives different series. For instance,

Input $x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots$ is a convergent geometric series ($r = -\frac{1}{3}$), or use Ratio Test

Input $x = 4$: $\sum_{n=0}^{\infty} \frac{(4)^n}{3^n} = 1 + \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \dots$ is a divergent geometric series ($r = \frac{4}{3}$), or use Ratio Test, or use Divergence Test.

What is the center of this series, $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$?

A. 0 **B.** 1 **C.** $\frac{1}{3}$ **D.** 3

$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n (x-0)^n$ so the center is 0.

Goal: For what interval of x-values does this power series converge? This interval is called the **interval of convergence**.

New vocab: the collection of inputs for which a power series converges.

THEOREM

For a given power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

there are only three possibilities

1. The series converges only when $x = a$
2. The series converges for all x
3. There is a positive number R such that …
    - the series converges if $|x-a| < R$ and
    - the series diverges if $|x-a| > R$.

   Note: if $|x-a| = R$, it could either converge or diverge. You must check!

What are the intervals of convergence in each of these situations?

1. converges for all x in $[a, a]$ interval is $\{a\}$
2. converges for all x in $(-\infty, \infty)$ interval is $(-\infty, \infty)$
3. converges for all $|x-a| < R$ i.e. for all x in $(a-R, a+R)$.
    - This might not be the interval of convergence!
    - Check whether $x = a - R$ is in the interval
    - Check whether $x = a + R$ is in the interval

Interval may be $(a-R, a+R)$ or $(a-R, a+R]$ or $[a-R, a+R)$ or $[a-R, a+R]$.

RADIUS OF CONVERGENCE

Memorize new vocab

The **radius of convergence** is the distance from the center of the series to one of the endpoints of the interval of convergence.

For our three possible types of intervals of convergence, the radii of convergence are as follows:

1. interval $[a, a]$ means radius is 0
2. interval $(-\infty, \infty)$ means radius is ∞
3. interval $(a-R, a+R)$ or $(a-R, a+R]$ or $[a-R, a+R)$ or $[a-R, a+R]$ means radius is R

Example: What is the *radius of convergence* for

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = 1 + \frac{x}{3} + \frac{x^2}{9} + \dots$$

Answer: $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$ is a geometric series with ratio $\frac{x}{3}$

So this series converges iff $\left|\frac{x}{3}\right| < 1 \iff |x| < 3$

$\iff -3 < x < 3$

Distance between the center (0) to one of the endpoints (3) is $\boxed{R = 3}$

(WEBWORK PROB 8, 16)

Use ideas about geometric series to find the interval of convergence and the radius of convergence for

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{5^n} = 1 + \frac{x+2}{5} + \frac{(x+2)^2}{25} + \dots$$

Answer:

$\sum_{n=0}^{\infty} \left(\frac{x+2}{5}\right)^n$ is a geometric series with ratio $\frac{x+2}{5}$

So the series converges iff $\left|\frac{x+2}{5}\right| < 1 \iff |x+2| < 5$

$$-5 < x+2 < 5$$
$$-5-2 < x < 5-2$$
$$-7 < x < 3$$

Interval of convergence is $(-7,3)$

Radius of convergence is $R=5$ (distance between center, 2, and an endpoint)

(WEBWORK PROB 8, 16, CONTINUED)

Use ideas about geometric series to find the interval of convergence and the radius of convergence for

OR, use Ratio Test:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{5^n} = 1 + \frac{x+2}{5} + \frac{(x+2)^2}{25} + \dots$$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum |a_n|$ is convergent

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum a_n$ is divergent.

Let $a_n = \left(\frac{x+2}{5}\right)^n$ $\quad \frac{a_{n+1}}{a_n} = \frac{(x+2)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(x+2)^n} = \frac{(x+2)^{n+1}}{\cancel{(x+2)^n}} \cdot \frac{\cancel{5^n}}{5^{n+1}} = \frac{x+2}{5}$

so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x+2}{5}\right| = \left|\frac{x+2}{5}\right|$

If $\left|\frac{x+2}{5}\right| < 1 \iff |x+2| < 5$, the series converges

If $\left|\frac{x+2}{5}\right| > 1 \iff |x+2| > 5$, the series diverges.

If $x+2=5$ or $x+2=-5$, we have $\sum_{n=0}^{\infty} 1$ and $\sum_{n=0}^{\infty} (-1)^n$ which are divergent.

WEBWORK PROB 5

Use the Ratio Test on the power series $\sum_{n=1}^{\infty} \frac{5^n(x-4)^n}{\sqrt{n}}$ Let $a_n = \frac{5^n(x-4)^n}{\sqrt{n}}$

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}(x-4)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{5^n(x-4)^n}$$

$$= 5\,\frac{\sqrt{n}}{\sqrt{n+1}}\,(x-4)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|5\,\frac{\sqrt{n}}{\sqrt{n+1}}\,(x-4)\right|$$

$$= |5(x-4)| \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= |5(x-4)|\cdot 1 \qquad \text{since } \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 1$$

By Ratio Test, $\sum a_n$ converges when $|5(x-4)| < 1$

$|x-4| < \frac{1}{5}$

$-\frac{1}{5} < x-4 < \frac{1}{5}$

WEBWORK PROB 5 CONTINUED

$$\sum_{n=1}^{\infty} \frac{5^n(x-4)^n}{\sqrt{n}} = 1 + (x-4) + \frac{(x-4)^2}{2} + \ldots$$

What is the center of the series? **A.** 0 [✓ **B.** 4] **C.** -4

What is the radius of convergence? **A.** 5 **B.** 10 [✓ **C.** $\frac{1}{5}$] **D.** $\frac{2}{5}$

What is the $\overbrace{\text{interval}}^{I}$ of convergence? We know I must include $(4-\frac{1}{5}, 4+\frac{1}{5})$

Check $x = 4-\frac{1}{5}$: $\sum_{n=1}^{\infty} \frac{5^n(-\frac{1}{5})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by Alternating Series Test (so I includes $x = 4-\frac{1}{5}$)

Check $x = 4+\frac{1}{5}$: $\sum_{n=1}^{\infty} \frac{5^n(\frac{1}{5})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2}$) (so I does not include $x = 4+\frac{1}{5}$)

$\boxed{I = [4-\frac{1}{5}, 4+\frac{1}{5})}$ is the interval of convergence

Think Factorial n! grows faster than exponential function $(\text{some number})^n$

so I expect $\sum x^n \frac{1}{n!}$ converges for all x

(WEBWORK PROB 10)

Find the $\underbrace{\text{radius}}_{R}$ and $\underbrace{\text{interval}}_{I}$ of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Let $a_n = \frac{x^n}{n!}$

Use Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{1}{n+1} \cdot x$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$ for any number x

So by Ratio Test, $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

$$\boxed{\begin{aligned} R &= \infty \\ I &= (-\infty, \infty) \end{aligned}}$$

What is the radius of convergence of this series? The center is 0.

A. 0 **B.** 1 **C.** 10 [✓ **D.** ∞]

Think Factorial n! grows faster than exponential function

so $|n!(x+8)^n| \to \infty$ unless $x+8=0$

so I expect $\sum n!(x+8)^n$ diverges except when $x=-8$

Similar practice

(WEBWORK PROB 9)

Find the radius and interval of convergence for $\displaystyle\sum_{n=0}^{\infty} n!(x+8)^n$.

Let $a_n = n!(x+8)^n$

Use Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{(n+1)!(x+8)^{n+1}}{n!(x+8)^n}$

$= (n+1)(x+8)$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |(n+1)(x+8)| = |x+8| \lim_{n\to\infty} (n+1) = \begin{cases} \infty & \text{if } x \neq -8 \\ 0 & \text{if } x = -8 \end{cases}$$

The series converges if and only if $x=-8$

$I = \{-8\}$, $R = 0$

What is the radius of convergence of this series?

[✓ **A.** 0] **B.** 1 **C.** 8 **D.** ∞

PRACTICE

ADDITIONAL PRACTICE I

Find the $\underbrace{\text{radius of convergence}}_{R}$ and $\overbrace{\text{the interval of convergence}}^{I}$ for

$$\sum_{n=0}^{\infty} 9^n (x-2)^{2n} = \sum_{n=0}^{\infty} \left[9(x-2)^2\right]^n$$

using geometric series.

This is a geometric series with ratio $9(x-2)^2$

So it converges if and only if $|9(x-2)^2| < 1$

$$|(x-2)^2| < \frac{1}{9}$$

$$|x-2| < \frac{1}{3}$$

$$-\frac{1}{3} < x-2 < \frac{1}{3}$$

$$2-\frac{1}{3} < x < 2+\frac{1}{3}$$

$R = \frac{1}{3}$ $\qquad$ $I = (1\frac{2}{3}, 2\frac{1}{3})$

What is the radius of convergence of this series?

A. 1 [✓ **B.** $\frac{1}{3}$] **C.** $\frac{1}{9}$ **D.** $\frac{1}{81}$

Since the center of this power series is 0, this means the radius of convergence, R, is between 3 and 4 (possibly 3 or 4).

PRACTICE

(WEBWORK PROB 6)

Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -3$ and diverges when $x = 4$.

Determine whether the following series converge or diverge.

C 1. $\sum_{n=1}^{\infty} c_n (1)^n$ Distance between center (0) and x=1 is $1<3$
$\Rightarrow$ the series converges

D 2. $\sum_{n=1}^{\infty} c_n 9^n$ Distance between center (0) and x=9 is $9>4$
$\Rightarrow$ the series diverges

C 3. $\sum_{n=1}^{\infty} c_n (-2)^n$ Distance between center (0) and x=-2 is $2<3$
$\Rightarrow$ the series converges

D 4. $\sum_{n=1}^{\infty} (-1)^n c_n 12^n = \sum_{n=1}^{\infty} c_n (-12)^n$ Distance between center (0) and x=-12 is $12>4$
$\Rightarrow$ the series diverges

PRACTICE

(WEBWORK PROB 2)

Find the radius of converge and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^4 x^{3n}}{3^{3n}} = \sum_{n=1}^{\infty} n^4 \left(\frac{x^3}{9}\right)^n$$

We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^4 x^{3(n+1)}}{3^{3(n+1)}} \cdot \frac{3^{3n}}{n^4 x^{3n}}\right| = \left(\frac{n+1}{n}\right)^4 \cdot \frac{x^3}{27}.$$

Since $(n+1)^4/n^4 \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^3}{27}.$$

We have $x^3/27 < 1$ when $|x| < 3$. The radius of convergence is 3 and the series converges for $-3 < x < 3$.

We check the endpoints. For $x = -3$, we have

$$\sum_{n=1}^{\infty} \frac{n^4 x^{3n}}{3^{3n}} = \sum_{n=1}^{\infty} \frac{n^4 (-3)^{3n}}{3^{3n}} = \sum_{n=1}^{\infty} (-1)^n n^4,$$

which diverges. Similarly, for $x = 3$, we have

by Test for Divergence

$$\sum_{n=1}^{\infty} \frac{n^4 x^{3n}}{3^{3n}} = \sum_{n=1}^{\infty} \frac{n^4 3^{3n}}{3^{3n}} = \sum_{n=1}^{\infty} n^4,$$

which diverges. The series diverges at both endpoints, so the interval of convergence is $-3 < x < 3$.

$(-3, 3)$

by Test for Divergence (or p-series with $p = -4$)

Radius of convergence : 3

Interval of convergence : $(-3, 3)$ since the power series diverges when $x = -3$, $x = 3$.