GROWTH RATES

Definition: Growth Rates of Functions (as x approaches infinity)

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then

- f *grows faster than* g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. or $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$
- f and g *have comparable growth rates* if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$, where M is a positive number.

Theorem: Asymptotic Hierarchy

Let the symbol $f \ll g$ mean that g grows faster than f as $x \to \infty$. Then

not covered in Math 2924

$$c \ll (\ln x)^q \ll x^p \ll a^x \ll \int_{t=0}^{\infty} t^x e^{-t}\,dt \ll x^x$$

$$\underset{\text{constant}}{c} \ll \underset{\text{logarithmic}}{(\ln n)^q} \ll \underset{\text{polynomial}}{n^p} \ll \underset{\text{exponential}}{a^n} \ll \underset{\text{factorial}}{n!} \ll n^n$$

p-sequence, geometric sequence, factorial

<u>Polynomials grow faster than logarithmic functions.</u>

Computation showing $\frac{x^p}{\ln(x)} \to \infty$ as $x \to \infty$: for positive p

$$\begin{aligned}\lim_{x\to\infty} \frac{x^p}{\ln x} &\overset{l'H}{=} \lim_{x\to\infty} \frac{p\,x^{p-1}}{\left(\frac{1}{x}\right)} \quad \text{by l'Hospital's Rule "}\tfrac{\infty}{\infty}\text{"}\\ &= \lim_{x\to\infty} p\,x^{p-1}x^1\\ &= \lim_{x\to\infty} p\,x^p\\ &= \infty \quad (\text{since } p \text{ is positive})\end{aligned}$$

<u>Exponential functions grow faster than polynomials.</u>

Computation showing that r^x (for $r > 1$) grows faster than x^p:

To save time, I'll do $r=4$ and $p=2$

$$\begin{aligned}\lim_{x\to\infty} \frac{4^x}{x^2} &\overset{l'H}{=} \lim_{x\to\infty} \frac{4^x \ln 4}{2x} \quad \text{by l'Hospital's Rule "}\tfrac{\infty}{\infty}\text{"}\\ &\overset{l'H}{=} \lim_{x\to\infty} \frac{4^x (\ln 4)(\ln 4)}{2} \quad \text{by l'Hospital's Rule "}\tfrac{\infty}{\infty}\text{"}\\ &= \infty\end{aligned}$$

Let $a=2$

$n=5$: a^5, $5!$, 5^5

$aaaaa < 5.4.3.2.1 < 5555$

$n=1000$: a^{1000}, $1000!$, 1000^{1000}

$\underbrace{aa\cdots a}_{1000 \text{ times}} < \underbrace{1000(999)(998)\ldots}_{1000 \text{ factors}} < \underbrace{(1000)(1000)\ldots}_{1000 \text{ times}}$

geometric sequence $\ll$ factorial $\ll n^n$

WHEN RATIO TEST SHOULD NOT BE USED

geometric sequence   factorial

$$c \ll (\ln n)^q \ll n^p \ll a^n \ll n! \ll n^n$$

Ratio Test will probably work if the terms have one of these

$$\lim_{n\to\infty} \frac{c}{c} = 1 \qquad \lim_{n\to\infty} \frac{(\ln(n+1))^q}{(\ln n)^q} = 1 \qquad \lim_{n\to\infty} \frac{(n+1)^p}{n^p} = 1$$

Ratio Test will not work if the terms only have logarithmic / polynomial-like terms

Ratio is comparing a_{n+1} with a_n as $n \to \infty$.

If $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$, we can't conclude anything about the series $\sum a_n$.

$$\boxed{\lim_{n\to\infty} \frac{c}{c}} = \lim_{n\to\infty} 1$$
$$= 1$$

To save time, I'll set $q = 1$

$$\boxed{\lim_{n\to\infty} \frac{(\ln(n+1))^q}{(\ln n)^q}} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)}$$

$$\overset{l'H}{=} \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} \quad \text{by l'Hospital's Rule } \text{“}\frac{\infty}{\infty}\text{”}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= 1$$

For example, $p=2$

$$\boxed{\lim_{n\to\infty} \frac{(n+1)^p}{n^p}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{1}$$

$$= 1$$

USING GROWTH RATES TO CHECK WHETHER A SERIES CONVERGES

Q: Will the Ratio Test work?

No, the Ratio Test will be inconclusive.

1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

(a) Fill in the blank with $\ll$ or $\gg$:

$$\ln(n+1) \; \underline{\ll} \; n.$$

(b) This means that

$$\lim_{n\to\infty} \frac{n}{\ln(n+1)} = \infty$$

(See page 1)

(c) Fill in the blank with $\ll$ or $\gg$:

$$\frac{1}{\ln(n+1)} \; \underline{\gg} \; \frac{1}{n}$$

(d) State whether the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges:

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u> because

<u>remember that the harmonic series is divergent</u> (or <u>$\sum \frac{1}{n}$ is a p-series with $p \leq 1$</u>)

(e) Let $a_n = \frac{1}{\ln(n+1)}$ and $b_n = \frac{1}{n}$. Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln(n+1)} = \infty$$

(f) Because of part $\boxed{e}$: $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

part $\boxed{d}$: $\sum b_n$ diverges

the series

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} \; \underline{\text{diverges}}$$

by <u>the Limit Comparison Test</u>

A COMBINATION OF TWO TYPES OF SERIES

2. Consider the series

$$\sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n} = \sum_{n=1}^{\infty} \frac{n^5}{\left(\frac{3}{2}\right)^n} . = \sum_{n=1}^{\infty} n^5 \left(\frac{2}{3}\right)^n$$

(a) The series $\sum_{n=1}^{\infty} n^5$ converges / diverges because $\sum_{n=1}^{\infty} \frac{1}{n^{-5}}$ is a p-series with $p \leq 1$ (or by Divergence Test).

(b) The series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges / diverges because it's a geometric series with ratio $\frac{2}{3}$ in $(-1, 1)$.

(c) Which one is more dominant? The numerator or the denominator?

Fill in the blank with $\ll$ or $\gg$:

see slide 1

$$n^5 \quad \underline{\ll} \quad \left(\tfrac{3}{2}\right)^n.$$

Since $\underline{\left(\frac{3}{2}\right)^n}$ is more dominant, I think $\sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n}$ will turn out to be $\underline{\text{Convergent}}$ like $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$.

(d) State the ratio test.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ is absolutely convergent.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ is divergent.

(e) Above, we said that if we see *only* logarithmic-like terms and polynomial-like terms, like $\ln(2n) + n^{p_1} + n^{p_2}$, the ratio test *will be inconclusive (Why?)*

But, if you see powers like a^n (or a factorial or n^n) then $\underbrace{\text{the ratio test will probably work}}$. So I think Ratio Test will work

Let $a_n = \frac{n^5 2^n}{3^n}$. Compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{\left(\frac{3}{2}\right)^{(n+1)}} \frac{\left(\frac{3}{2}\right)^n}{n^5}$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \frac{\left(\frac{3}{2}\right)^n}{\left(\frac{3}{2}\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \frac{1}{\left(\frac{3}{2}\right)}$$

$$= \lim_{n\to\infty} \frac{(n^5 + \ldots)}{n^5} \frac{2}{3}$$

$$= 1 \cdot \frac{2}{3} < 1$$

(f) By the ratio test, the series

$$\sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n} \quad \underline{\text{converges}}.$$

A COMBINATION OF MORE THAN TWO TYPES OF SERIES

3. Consider the series

$$\sum_{n=1}^{\infty} a_n$$

for

$$a_n = \frac{n^n}{7^n\,(n)!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\,(n)!}$$

(a) Look at the term of the series.

The numerator is n^n the fastest growing sequence in our asymptotic hierarchy (slide 1)

The denominator, $7^n\,(n)!$ is the product of a geometric sequence and factorial $n!$.

Hierarchy: $7^n \ll n! \ll n^n$

(b) Which do you think is more dominant for large n? The $\underbrace{\text{numerator}}_{n^n}$ or the $\underbrace{\text{denominator}}_{a^n\,n!}$?

(c) I told you that the ratio test *will probably work* if you see a geometric sequence (r^n) or a factorial or n^n. Compute → So try Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

for

$$a_n = \frac{n^n}{7^n\,n!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\,n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{(n+1)^{n+1}}{7^{n+1}(n+1)!}\right)}{\left(\frac{n^n}{7^n n!}\right)}$$

$$= \frac{(n+1)^{n+1}}{7^{n+1}(n+1)!}\cdot\frac{7^n\,n!}{n^n}$$

$$= \frac{(n+1)^{n+1}}{n^n}\cdot\frac{\cancel{7^n}}{7^{\cancel{n+1}}}\cdot\frac{\cancel{n!}}{\cancel{(n+1)!}}$$

$$= \frac{(n+1)^{n+1}}{n^n}\;\frac{1}{7}\;\frac{1}{(n+1)}$$

$$= \frac{(n+1)^n}{n^n}\;\frac{1}{7}$$

$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = ?$ Indeterminate form, type "1^∞"

Let $y = \left(1+\frac{1}{x}\right)^x$

$\ln y = \ln\left[\left(1+\frac{1}{x}\right)^x\right] = x\ln\left[1+\frac{1}{x}\right]$

$\lim\limits_{x\to\infty}\ln y = \lim\limits_{x\to\infty}\dfrac{\ln\left[1+\frac{1}{x}\right]}{\left(\frac{1}{x}\right)}$

$\overset{l'H}{=} \lim\limits_{x\to\infty}\dfrac{\frac{1}{\left(1+\frac{1}{x}\right)}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$ by l'Hospital's Rule "$\frac{0}{0}$"

$= 1$

So $\lim\limits_{x\to\infty} y = e^{\left(\lim_{x\to\infty}\ln y\right)} = e^1$

(Recall: $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$ using l'Hospital's Rule for indeterminate powers, Sec 6.8)

(d) By the ratio test, the series

$$\sum \frac{n^n}{7^n\,n!} \text{ converges} \qquad \text{and} \qquad \sum \frac{n^n}{2^n\,n!} \text{ diverges}.$$

$\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \dfrac{e}{7} < 1$

$\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \dfrac{e}{2} > 1$

The number e is between 2 and 3

Use either Limit Comparison Test or Alternating Series Test to determine

whether $\displaystyle\sum_{n=3}^{\infty} \frac{(n^2-6)\cos(n\pi)}{n^5}$ converges / diverges

From Webwork 11.7

Answer: $\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

so $\displaystyle\sum_{n=3}^{\infty} \frac{(n^2-6)\cos(n\pi)}{n^5} = \sum_{n=3}^{\infty} \frac{(n^2-6)(-1)^n}{n^5} = \sum_{n=3}^{\infty} (-1)^n b_n$ for $b_n := \dfrac{n^2-6}{n^5}$

Since $b_n > 0$ for all $n \geq 3$,

b_n is decreasing, and

$\displaystyle\lim_{n\to\infty} b_n = 0$,

the series converges by Alternating Series Test

Webwork 11.7 Prob 1

Use the Limit Comparison Test to show that $\sum_{n=1}^{\infty} \frac{\ln(2n) + 3n}{n^2}$ diverges.

Answer: Let $a_n = \frac{\ln(2n) + 3n}{n^2}$

Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(2n) + 3n}{n^2} \cdot n$$

$$= \lim_{n\to\infty} \frac{\ln(2n)}{n} + 3$$

$$= 0 + 3 = 3$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p \leq 1)$

By the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n$ also diverges.