Review, copied from lecture Sec 11.5 (pg 3)

① Is $\sum_{n=1}^{\infty} n^2\left(-\frac{1}{e}\right)^n$ an alternating series?

$\sum (-1)^n \frac{n^2}{e^n}$ always positive

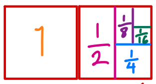
② Evaluate $\lim_{n\to\infty} \frac{n^2}{e^n} = \lim_{n\to\infty} \frac{2n}{e^n} = \lim_{n\to\infty} \frac{2}{e^n} = 0$

l'H "$\frac{\infty}{\infty}$"   l'H "$\frac{\infty}{\infty}$"

1. **Recall** The geometric series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ converges to $\frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}$

Without $(-1)^n$

2. The geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges to $\frac{1}{1-\left(\frac{1}{2}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$



converges to the area of this 1x2 rectangle

3. The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the Alternating Series Test (Sec 11.5)

since (i) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing and (ii) $\lim_{n\to\infty} \frac{1}{n} = 0$

Remove $(-1)^n$

4. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series theorem $(p=1)$ or

remember that the harmonic series diverges

Above examples illustrate that removing the alternating signs in a convergent series may or may not result in a convergent series. Below terminology distinguishes these cases.

Absolute and Conditional Convergence

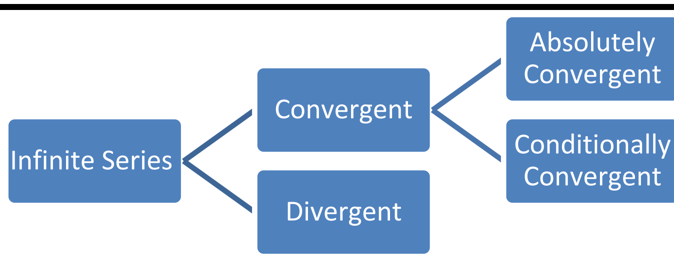
Memorize New Vocab

Definition Absolute and Conditional Convergence

Assume the infinite series $\sum a_n$ converges. The terms a_n may/may not be all positive or all negative (no restriction)

1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the series $\sum a_n$ **converges absolutely**.

2. If $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that the series $\sum a_n$ **converges conditionally**.



5. Textbook Example 2: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges by Alternating Series Test

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2)$,

so we say $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

Sample answer to follow

6 Example:

Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ diverges, converges absolutely, or converges conditionally.

since (i) $\left\{ \frac{1}{\sqrt{n}} \right\}_{n=1}^{\infty}$ is decreasing

(ii) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$,

the series converges by Alternating Series Test.

But $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent p-series $\left(p = \frac{1}{2}\right)$

So we say $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent.

Sample answer to follow

TASK. Copy the solution to Textbook Example 1, pg 777 ← Similar example

If a series is absolutely convergent, then it is convergent.

Theorem 3 **Absolute Convergence Implies Convergence**

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof Idea:

$$\sum -|a_n| \leq \sum a_n \leq \sum |a_n|$$

Example:

Determine whether $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ diverges, converges absolutely, or converges conditionally.

$\sum \frac{\sin n}{n^2}$ has both positive and negative terms,

but $\sum \frac{\sin n}{n^2}$ is not an alternating series (so Alternating Series Test doesn't apply)

(We can apply comparison Test to $\sum \left|\frac{\sin n}{n^2}\right|$ with $\sum \frac{1}{n^2}$)

Let $a_n := \frac{|\sin n|}{n^2}$ and $b_n := \frac{1}{n^2}$

Since (i) $0 \leq a_n \leq b_n$ for all $n = 1, 2, 3, \ldots$ and

(ii) $\sum b_n$ is a convergent p-series $(p = 2)$,

$\sum \left|\frac{\sin n}{n^2}\right|$ also converges by the Comparison Test.

By def, $\sum \frac{\sin n}{n^2}$ absolutely converges.

← Sample answer to follow

(meaning $\sum \frac{\sin n}{n^2}$ converges and $\sum \left|\frac{\sin n}{n^2}\right|$ converges).

TASK. Copy the solution to Textbook Example 3, pg 778 ← Similar example

Definition **Factorial**

The **factorial** of a positive integer n, denoted by $n!$, is the **product** of all positive integers less than or equal to n.

- Simplify $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.
- $0! \overset{def}{=} 1$.
- Simplify $\dfrac{(n+1)!}{n!} = \dfrac{(n+1)\cancel{(n)(n-1)\cdots 2\cdot 1}}{\cancel{n(n-1)\cdots 2\cdot 1}} = n+1$

Theorem **The Ratio Test** Memorize!

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series Consider $r := \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$.

- (i) If $0 \le r < 1$, then $\underbrace{\sum_{n=1}^{\infty} |a_n| \text{ is convergent}}_{\text{called "absolutely convergent"}}$.
- (ii) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) $r = 1$, then the Ratio Test is inconclusive

Example: Use the **Ratio Test** to determine whether the series $\sum_{k=1}^{\infty}\dfrac{10^k}{k!}$ converge.

$$\frac{a_{k+1}}{a_k} = \frac{\left(\frac{10^{k+1}}{(k+1)!}\right)}{\left(\frac{10^k}{k!}\right)} = \frac{10^{k+1}}{(k+1)!}\cdot\frac{k!}{10^k} = \frac{10}{k+1}$$

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{10}{k+1} = 0$$

Sample answer

$\sum_{k=1}^{\infty}\dfrac{10^k}{k!}$ is convergent / divergent by the Ratio Test, since $\lim_{k\to\infty}\dfrac{a_{k+1}}{a_k} = 0 < 1$

(absolutely convergent)

Practice with more examples

TASK. Practice applying the Ratio Test to conclude absolute convergence. Copy the solution to Textbook Example 4, pg 780

EXAMPLE 4 Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$: . . .

TASK. Practice applying the Ratio Test to conclude divergence. Copy the solution to Textbook Example 5, pg 781

EXAMPLE 5 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs. . . .

■

Practice with more examples

TASK. Practice applying the Ratio Test to conclude absolute convergence. Copy the solution to Textbook Example 4, pg 780

EXAMPLE 4 Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent. ■

TASK. Practice applying the Ratio Test to conclude divergence. Copy the solution to Textbook Example 5, pg 781

EXAMPLE 5 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e \qquad \text{as } n \to \infty$$

See Exam 1 sol for Sec 6.8 l'Hospital's Rule

(see Equation 6.4.9 or 6.4*.9). Since $e > 1$, the given series is divergent by the Ratio Test. ■

The Root Test

Theorem **The Root Test**

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series. Consider $r = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

- If $0 \le r < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (& so is convergent).
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- $r = 1$, the Root Test is inconclusive.

Example: Use the **Root Test** to determine whether the series $\sum_{k=1}^{\infty} \left(\frac{2k+3}{3k+2}\right)^k$ converges.

Let $a_k := \left(\frac{2k+3}{3k+2}\right)^k$

$$\sqrt[k]{|a_k|} = \sqrt[k]{\left(\frac{2k+3}{3k+2}\right)^k} = \frac{2k+3}{3k+2}$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{2k+3}{3k+2} = \frac{2}{3} < 1$$

So $\sum a_k$ is absolutely convergent by the Root Test.
(hence convergent)

- The **Root Test** is inconclusive implies the **Ratio Test** is inconclusive.

and vice versa

(So if $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$, $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ will be 1 also)