Lecture 11.5 Alternating Series

A series is called alternating if the terms are alternately positive and negative.

Thm

Alternating Series Test.

IF the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\, b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \qquad \text{where all } b_n > 0$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}\, b_n = -b_1 + b_2 - b_3 + b_4 - b_5 + b_6 - \cdots, \qquad \text{where all } b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n,

(ii) $\lim_{n\to\infty} b_n = 0$,

THEN

the series converges.

(Justification is given on Textbook pg 773)

Note: If either condition in this test fails then the test cannot be used.

Example of an alternating series

Example: Decide if $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges or diverges.

Thinking about the problem:

The series starts off as $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ and is alternating with $b_n = \frac{1}{2n-1}$. Let's check the conditions for the Alternating Series Test.

Doing the problem:

For $b_n = \frac{1}{2n-1} > 0$ we need to check (i.) $\overbrace{b_{n+1} \leq b_n \text{ for all } n}^{\text{decreasing}}$ and (ii.) $\overbrace{b_n \to 0 \text{ as } n \to \infty}^{b_n \text{ goes to } 0}$.

(i.) The inequality $b_{n+1} \leq b_n$ is the same as $\overbrace{\frac{1}{2n+1}}^{b_{n+1}} \leq \overbrace{\frac{1}{2n-1}}^{b_n}$, which is equivalent to saying $2n+1 \geq 2n-1$, and that last inequality is true. Alternatively, using calculus, the function $f(x) = \frac{1}{2x-1}$ has derivative $f'(x) = -\frac{2}{(2x-1)^2}$, which is negative for $x \geq 1$, so $f(x)$ is decreasing for $x \geq 1$.

(ii.) For the limit, $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$.

We can conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ __converges__ using the Alternating Series Test.

Note: The Alternating Series Test can never be used to determine divergence. If you think an alternating series is divergent, use another thm to show this.

1. Determine if the series $\sum_{n=1}^{\infty} n\left(-\frac{1}{2}\right)^n$ converges or diverges, with justification.

   (a) Decide what b_n should be, and check that the series fits the conditions of the Alternating Series Test.

   (b) Using the Alternating Series Test, does the series converge or diverge?

Rewrite $\sum_{n=1}^{\infty} n(-1)^n\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}(-1)^n \frac{n}{2^n}$

Recall: $b^x = e^{x\ln b}$

so this is an alternating series with $b_n = n\left(\frac{1}{2}\right)^n$

$\frac{d}{dx}b^x = \ln b\, e^{x\ln b}$

(i) Let $f(x) = x\left(\frac{1}{2}\right)^x$

$$f'(x) = 1\left(\tfrac{1}{2}\right)^x + x\,\ln\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^x$$
$$= \left(\tfrac{1}{2}\right)^x\left[1 + x\ln\left(\tfrac{1}{2}\right)\right]$$
$$= \left(\tfrac{1}{2}\right)^x\left[1 - x\left(\ln 1 - \ln 2\right)\right]$$
$$= \left(\tfrac{1}{2}\right)^x\left[1 - x\ln 2\right]$$

which is negative if $1 - x\ln(2) < 0$

$$1 < x\ln(2)$$
$$1.4 \approx \frac{1}{\ln(2)} < x$$

Recall: $0 = \ln 1 < \ln(2) < \ln(e) = 1$

So $b_{n+1} \le b_n$ for all $n = 2, 3, 4, \ldots$ and condition (i) is satisfied

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{2^n} \overset{\text{l'Hospital's Rule "}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty}\frac{1}{2^n\ln 2} = 0$ so condition (ii) is satisfied.

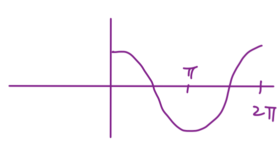
By the Alternating Series Test, $\sum_{n=1}^{\infty} n\left(-\frac{1}{2}\right)^n$ converges.

Webwork

2. True or false? The infinite series $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{1/4}}$ is alternating.

$-\frac{1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \frac{-1}{5} + \ldots$

$$\cos(n\pi) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}\end{cases}$$



so $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{1/4}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{1/4}}$ is alternating with $b_n = \frac{1}{n^{1/4}}$.

Helpful Examples from Textbook

3. Determine if the following series converge or diverge, with justification.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (Copy solution from Example 1, pg 774)

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ (Copy solution from Example 2, pg 774)

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{3n}{4n-1}$ (Copy solution from Example 2, pg 774)