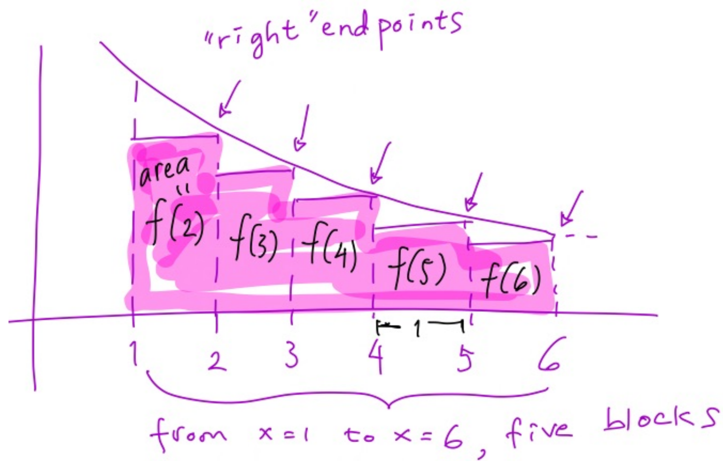
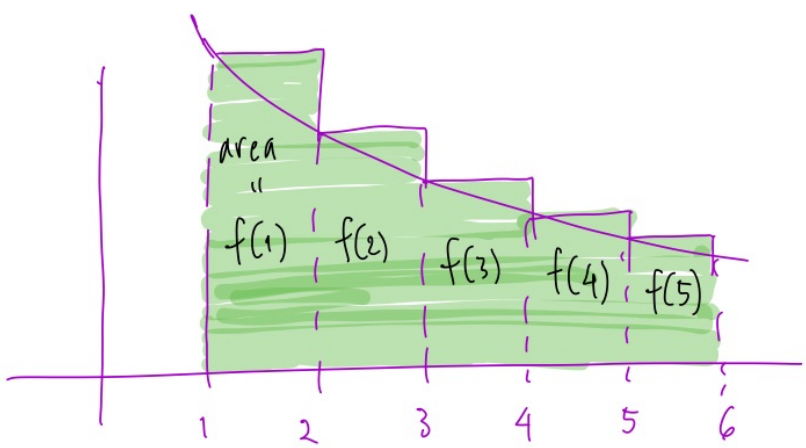
Intuition for the Integral Test:

Suppose $f(x)$ is a continuous and positive function on $[1,\infty)$.

a. Use the **Right Endpoint Rule** with $n=5$ to approximate the integral $\int_1^6 f(x)\,dx$.



b. Use the **Left Endpoint Rule** with $n=5$ to approximate the integral $\int_1^6 f(x)\,dx$.



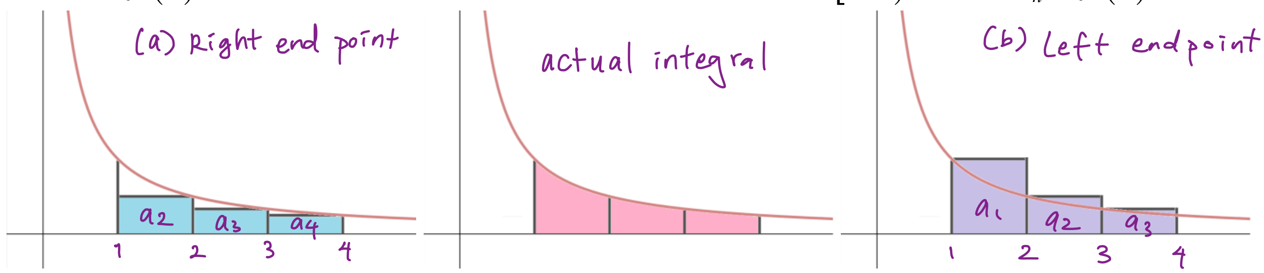
c. Suppose $f(x)$ is **decreasing**, then

the estimated value in part (a) $\underline{\ \leq\ }$ the value of $\int_1^6 f(x)\,dx$ and

the estimated value in part (b) $\underline{\ \geq\ }$ the value of $\int_1^6 f(x)\,dx$.

Integral Test

Suppose $f(x)$ is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then



$$a_2 + a_3 + a_4 \quad \le \quad \int_1^4 f(x)\,dx \quad \le \quad a_1 + a_2 + a_3$$

$$a_2 + \dots + a_6 \quad \le \quad \int_1^6 f(x)\,dx \quad \le \quad a_1 + a_2 + \dots + a_5$$

In general,

$$\underbrace{\sum_{k=2}^{n} a_k}_{(a)} \le \int_1^n f(x)\,dx \le \underbrace{\sum_{k=1}^{n-1} a_k}_{(b)}$$

The Integral Test

Suppose f is a **continuous**, **positive**, **decreasing** function on $[1,\infty)$ and let $a_n = f(n)$. Then

- If $\int_1^\infty f(x)\,dx$ is **convergent**, then $\sum_{n=1}^{\infty} a_n$ is Convergent as well.
- If $\int_1^\infty f(x)\,dx$ is **divergent**, then $\sum_{n=1}^{\infty} a_n$ is divergent as well

When we use the Integral Test

- It is not necessary to start the series or the integral at $n=1$. For example, in testing the series $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ we can use $\int_4^\infty \frac{1}{(x-3)^2}\,dx$.

  e.g. OK to start at $n=4$
- It is not necessary that f be **always** decreasing. What is important is that f be **ultimately** decreasing. That is, decreasing on $[N,\infty)$ for some number N. Then $\sum_{n=N+1}^{\infty} a_n$ is convergent, which means $\sum_{n=1}^{\infty} a_n$ is convergent.

We should **NOT** infer from the Integral Test that the sum of the series is equal to the value of the integral. In general,

$$\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{often difficult}} \ne \underbrace{\int_1^\infty f(x)\,dx}_{\text{often easier}}.$$

Example: Suppose we know that

- f is continuous, positive, and decreasing on $[2, \infty)$, and
- If $t > 2$, then $\displaystyle\int_2^t f(x)\,\mathrm{d}x = \frac{1}{\ln 2} - \frac{1}{\ln t}$.

Use the Integral Test (above) to determine whether the series $\displaystyle\sum_{k=2}^{\infty} f(k)$ converges or diverges.

Answer First step (Check whether $\int_2^\infty f(x)\,\mathrm{d}x$ converges or diverges.)

$$\int_2^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{t\to\infty} \int_2^t f(x)\,dx = \lim_{t\to\infty} \frac{1}{\ln 2} - \frac{1}{\ln t} = \frac{1}{\ln 2}$$

So $\int_2^\infty f(x)\,dx$ converges

Second step:

- If $\int_2^\infty f(x)\,\mathrm{d}x$ converges, then $\displaystyle\sum_{k=2}^{\infty} f(k)$ also converges by the Integral Test.
- If $\int_2^\infty f(x)\,\mathrm{d}x$ diverges, then $\displaystyle\sum_{k=2}^{\infty} f(k)$ also diverges by the Integral Test.

Since $\int_2^\infty f(x)\,dx$ converges, $\displaystyle\sum_{k=2}^{\infty} f(k)$ also converges.

Question: Does this mean that $\displaystyle\sum_{k=2}^{\infty} f(k) = \frac{1}{\ln 2}$? NO. In general, $\displaystyle\sum_{k=2}^{\infty} f(k) \neq \int_2^\infty f(x)\,dx$

Example: Suppose we know that

- g is continuous, positive, and decreasing on $[1, \infty)$, and
- If $t > 1$, then $\displaystyle\int_1^t g(x)\,\mathrm{d}x = 2\sqrt{t+5} - 2\sqrt{6}$.

Use the Integral Test (above) to determine whether the series $\displaystyle\sum_{k=1}^{\infty} g(k)$ converges or diverges.

Answer First step:

$$\int_1^\infty g(x)\,dx \overset{\text{def}}{=} \lim_{t\to\infty} \int_1^t g(x)\,dx = \lim_{t\to\infty} \left(2\sqrt{t+5} - 2\sqrt{6}\right) = \infty$$

So $\int_1^\infty g(x)\,dx$ diverges

Second step:

Since $\int_1^\infty g(x)\,dx$ diverges, $\displaystyle\sum_{k=1}^{\infty} g(k)$ also diverges.

Example (Copy the solution from Example 4 (page 570 from Sec 7.8 Improper Integral)

For what values of p is the improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

$\boxed{\text{Answer: convergent iff } p > 1}$

Why?

If $p \neq 1$, then if $t > 1$, $\displaystyle\int_1^t \frac{1}{x^p}\,dx = \int_1^t x^{-p}\,dx$

$$= \left.\frac{x^{-p+1}}{-p+1}\right|_{x=1}^{x=t}$$

$$= \frac{t^{-p+1}-1}{-p+1}$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx$$

$$= \lim_{t\to\infty}\frac{t^{-p+1}-1}{-p+1} = \begin{cases} \dfrac{0-1}{-p+1} & \text{if } -p+1 \text{ is negative} \\ \infty & \text{if } -p+1 \text{ is positive} \end{cases}$$

$t^{(\text{negative number})} \to 0$ as $t \to \infty$

$t^{(\text{positive number})} \to \infty$ as $t \to \infty$

Note: $-p+1$ is negative $\Leftrightarrow$ $p > 1$

and

$-p+1$ is positive $\Leftrightarrow$ $p < 1$

Task (Copy the solution from Example 1, page 568 from Sec 7.8): Evaluate

Do at 

$$\int_1^\infty \frac{1}{x^p}\,dx$$

when $p = 1$

$\boxed{\text{Answer: Diverges when } p = 1}$

Why?

$$\int_1^\infty \frac{1}{x}\,dx = \ldots$$

$$= \infty$$

Convergence and Divergence of the p-series

If $p \le 0$, $\sum \frac{1}{n^p}$ diverges by Test for Divergence

For any number p, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series.

If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$. If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.

In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the Test for Divergence.

If $p > 0$, use the Integral Test to check convergence/divergence

If $p > 0$, then the function $f(x) = \frac{1}{x^p}$ is continuous, positive and decreasing on $[1,\infty)$.

Previous slide: $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \le 1$.

So, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$ by the Integral Test.

***p*-series**

To remember this :

* $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent}}$
* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ also diverges if $\frac{1}{n^p} \geq \frac{1}{n}$
* Other p-series converges.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

convergent if $p > 1$ and divergent if $p \le 1$.

In particular, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Practice/Review: Determine whether the series $\sum_{k=1}^{\infty} k^{-\frac{3}{4}}$ converges or diverges.

$\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{4}}}$, $p = \frac{3}{4} < 1$, so $\frac{1}{k^{3/4}} > \frac{1}{n}$, so $\sum \frac{1}{k^{3/4}}$ diverges

Practice/Review: Determine whether the series $\sum_{k=4}^{\infty} \frac{1}{(k-1)^{\sqrt{2}}}$ converges or diverges.

First term is $\frac{1}{(4-1)^{\sqrt{2}}} = \frac{1}{3^{\sqrt{2}}}$ → $\sum_{k=3} \frac{1}{k^{\sqrt{2}}}$ → so $p = \sqrt{2} > 1$

Practice/Review: Which of the following is a convergent p-series?

A.) $\sum_{k=1}^{\infty} \frac{3}{2^k}$ $r = \frac{1}{2}$

B.) $\sum_{k=1}^{\infty} \frac{3}{\left(\frac{1}{2}\right)^k}$ $r = 2^k$

$\underbrace{}_{\text{Geometric series}}$

A: convergent, but not a p-series

B: Not convergent. Also not a p-series

C.) $\sum_{k=1}^{\infty} \frac{3}{k^2}$

* p-series $p = 2 > 1$
* Convergent

D.) $\sum_{k=1}^{\infty} \frac{3}{k^{\frac{1}{2}}}$

p-series $p = \frac{1}{2} < 1$ not convergent

Possible Strategy (so far)

Assume $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n > 0$ for all n.

1. Check if it is a **Geometric Series**.
   No! Go to (2).
   Yes! If $r \geq 1$ or $r \leq -1$, then the series diverges. If $-1 < r < 1$, then $S = \frac{a_1}{1-r}$.

2. Check if it is a **p-Series**.
   No! Go to (3).
   Yes! If $p \leq 1$, then the series diverges. If $p > 1$, then the series converges.

3. Check if $\lim_{k \to \infty} a_k = 0$. (**L'Hôpital's Rule is used if necessary**)
   Yes! Then the test is inconclusive. Go to (4).
   No! Then the series diverges by the **Test for Divergence**.

4. Check if it is a **Telescoping Series**.
   No! Go to (5).
   Yes! Evaluate S_n by cancelling middle terms and $S = \lim_{n \to \infty} S_n$.

5. Use the following Tests:
   The Integral Test (when a_n is positive, decreasing, and "easy to integrate")
   More tests to come

Extra practice questions:
Use one of the above methods to determine whether the following series converge.

Divergent a) $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$ Geometric series ratio is $\left(\frac{1}{\ln 2}\right)^n$

$\ln 2 < \ln e = 1$ so $1 < \frac{1}{\ln 2}$

Divergent by Test for Divergence b) $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ Neither a geometric series nor p-series

$\lim_{n\to\infty} \frac{2^n}{n+1} = \lim_{n\to\infty} \frac{(\ln 2)\, 2^n}{1} = \infty$

Convergent c) $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$ p-series $p = \frac{3}{2} > 1$

Divergent d) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n)$ Telescoping series –

See page 4 of Lecture 11.2 note