Infinite Series

If we add the terms of a sequence $\{a_k\}_{k=1}^{n}$, we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n \qquad a_1 + a_2 + a_3 \qquad a_2 + a_3 + a_4 + a_5$$

which is called a (finite) **series** and is also denoted by

$$\sum_{k=1}^{n} a_k. \qquad \sum_{k=1}^{3} a_k \qquad \sum_{k=2}^{5} a_k$$

specify starting index

Does it make sense to talk about the sum of infinitely many terms? Consider the **partial sums**

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3,$$

and, in general,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

"First, take the sum of the first n terms"

If the sequence $\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, S_3, \dots\}$ of partial sums has limit L, then we say that the infinite series **converges** to L and we write

$$\lim_{n\to\infty} S_n = L \quad \xleftrightarrow{\text{same meaning}} \quad \sum_{k=1}^{\infty} a_k = L$$

If the sequence $\{S_n\}_{n=1}^{n}$ of partial sums diverges, then we say that the infinite series **diverges**.

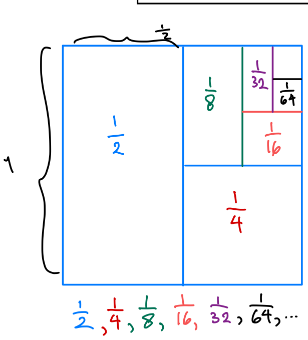
Summary(Notation)

• A **sequence** converges? $\lim_{k\to\infty} a_k = L$ (a number), $\{a_k\}$ converges	A **sequence** diverges? $\lim_{k\to\infty} a_k$ doesn't exist, $\{a_k\}$ diverges
• A **series** converges? $\lim_{n\to\infty} S_n = L$, $\sum_{k=1}^{\infty} a_k = \underbrace{L}_{\text{a number}}$, $\sum_{k=1}^{\infty} a_k$ converges	A **series** diverges? $\lim_{n\to\infty} S_n$ doesn't exist, $\sum_{k=1}^{\infty} a_k$ diverges

An important family of infinite series is the geometric series.

Visual example

1x1 square



$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \dots$

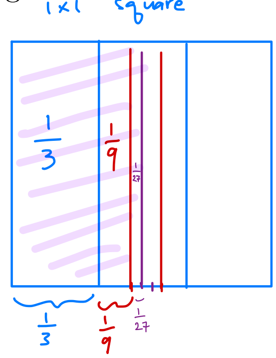
Area of 1x1 square is 1

So it seems like

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

converges to 1

1x1 square



The highlighted region seems to approach half of the square.

So $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$ seems to converge to $\frac{1}{2}$

Recall

- A **geometric sequence** has the property that each term is obtained by multiplying the previous term by a fixed constant, called the **ratio**, e.g. $\underbrace{\{5, 10, 20, 40, 80, 160, \ldots\}}_{\text{ratio}=2}$.
- Given a geometric sequence $\{a_k\}_{k=1}^{\infty}$, if the ratio is r, then the k-th term can be expressed as $a_k =$ $\underbrace{a_1}_{\text{first term}} r^{k-1}$, e.g. $a_k = 5 \; 2^{k-1}$ for $k = 1, 2, 3, \ldots$
- When $-1 < r \leq 1$, the sequence converges.

Geometric Series

Partial Sum of Geometric Series (Textbook Example 2)

Given a geometric sequence $\{a_k\}_{k=1}^{\infty}$, if the ratio is r, then the sum of the first n terms

$$S_n \overset{def}{=} a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$$

e.g. $S_4 = a_1 + a_1 r + a_1 r^2 + a_1 r^3$

is (see below) .

Why?

$$\begin{array}{rcccccccc} S_n = & a_1 + & a_1 r + & a_1 r^2 + & a_1 r^3 + & \ldots + & a_1 r^{n-2} + & a_1 r^{n-1} & \\ r\,S_n = & 0 & a_1 r + & a_1 r^2 + & a_1 r^3 + & \ldots + & a_1 r^{n-2} + & a_1 r^{n-1} + & a_1 r^n \end{array} \; -$$

$S_n - r S_n = a_1 + 0 + 0 + \cdots 0 - a_1 r^n$

Therefore, $S_n - r\,S_n = a_1 - a_1 r^n$,

$S_n (1 - r) = a_1 (1 - r^n)$

hence $S_n = \dfrac{a_1 (1 - r^n)}{1 - r}$ if $r \neq 1$.

Furthermore, since

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{for } |r| < 1 \\ \text{DNE} & \text{for } r = -1 \\ \text{DNE} & \text{for } |r| > 1 \\ 1 & \text{for } r = 1 \end{cases}, \text{ we have } \lim_{n\to\infty} S_n = \lim_{n\to\infty} a_1 \frac{1-r^n}{1-r} = \begin{cases} a_1 \frac{1-0}{1-r} = a_1 \frac{1}{1-r} & \text{for } |r| < 1 \\ \text{DNE} & \text{for } r = -1 \\ \text{DNE} & \text{for } |r| > 1 \\ \text{DNE } (\infty) & \text{for } r = 1 \end{cases},$$

The partial sums $\{S_n\}$ do not converge if $|r| \geq 1$

Theorem (Geometric Series)

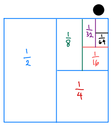
Let r and a be real numbers.

If $|r| < 1$, then $\sum_{k=1}^{\infty} ar^{k-1}$ $= a \frac{1}{1-r}$.

If $|r| \geq 1$, then $\sum_{k=1}^{\infty} ar^{k-1}$ diverges.

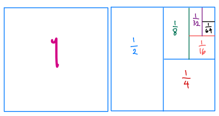
Note: In general, it's very hard to compute sums of (convergent) series

- The **geometric sequence** converges if and only if $-1 < r \leq 1$ (including 1).
- The **geometric series** converges if and only if $-1 < r < 1$ (not including 1).
- **Sum of a convergent series may change if you change your starting index:**



$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \overset{\text{slide 1}}{=}$ 1, the area of a 1×1 square. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k =$

1st term



$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = 1 + 1 = 2$, the area of a 1×2 rectangle.

first term

If $|r| < 1$, then $\sum_{k=1}^{\infty} ar^{k} = ar + ar^2 + ar^3 + \ldots = ar\left[1 + r + r^2 + \ldots\right] = ar \frac{1}{1-r}$

by above Thm (Geometric Series)

Example: Evaluate the (geometric) series $\sum_{k=1}^{\infty} \frac{3^k}{4^{k+3}}$ or state that it diverges.

step a.) State the test you plan to use: Geometric Series Thm (above)

step b.) (i) Write out the first 4 (four) terms of $\sum_{k=1}^{\infty} \frac{3^k}{4^{k+3}}$.

$\frac{3^1}{4^4}$ $\frac{3^2}{4^5}$ $\frac{3^3}{4^6}$ $\frac{3^4}{4^7}$ Note: To go from $\frac{3^2}{4^5}$ to $\frac{3^3}{4^6}$, multiply by $\frac{3}{4}$

(ii) Write our the first 4 (four) terms of $\sum_{k=1}^{\infty} ar^{k-1}$.

a ar^1 ar^2 ar^3 Note: To go from ar^1 to ar^2, multiply by r

(iii) Compare terms to find an a and an r so that $\sum_{k=1}^{\infty} \frac{3^k}{4^{k+3}} = \sum_{k=1}^{\infty} ar^{k-1}$.

$a = \frac{3}{4^4}$ $r = \frac{3}{4}$

step c.) After finding the ratio r, determine whether this geometric series converges or not.

Since $r = \frac{3}{4}$ is in $(-1, 1)$, the series converges to $a\frac{1}{1-r} = \frac{3}{4^4}\left(\frac{1}{1-\frac{3}{4}}\right) = \frac{3}{4^3}$

Task: (Follow the above example or copy solution from Textbook Example 4, pg 750.)
Express $\sum_{k=1}^{\infty} 2^{2k} 3^{1-k}$ as a geometric series, then determine whether it is convergent or divergent.

Do at 

Telescoping Series

In general, $\sum_{k=1}^{n} f(k) - f(k+4) = f(1)+f(2)+f(3)+f(4) - f(n+1) - f(n+2) - f(n-3) - f(n-4)$

Webwork: Evaluate the series $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+4}\right)$ or state that it diverges.

step a.) Find a formula for the k-th term of the sequence of **partial sums** $\{S_n\}$

$$S_n = \sum_{k=1}^{n} \ln(k) - \ln(k+4)$$

$$= \underbrace{\ln(1)}_{0} - \ln(\overset{1+4}{5}) + \ln(2) - \ln(\overset{2+4}{6}) + \ln(3) - \ln(\overset{3+4}{7}) + \ln(4) - \ln(\overset{4+4}{8})$$

$$+ \ln(5) - \ln(\underset{5+4}{9}) + \ln(6) - \ln(\underset{6+4}{10}) + \ln(7) - \ln(\underset{7+4}{11}) + \ln(8) - \ln(\underset{8+4}{12})$$

$$+ \cdots$$

$$-\ln(n-1) \quad + \ln(\overset{k=n-4}{n-4}) - \ln(n) + \ln(\overset{k=n-3}{n-3}) - \ln(n+1) + \ln(\overset{k=n-2}{n-2}) - \ln(n+2)$$

$$+ \ln(\underset{k=n-1}{n-1}) - \ln(n+3) + \ln(\underset{k=n}{n}) - \ln(n+4)$$

$$= \underbrace{\ln(2) + \ln(3) + \ln(4)}_{\ln(2\cdot 3\cdot 4)} - \ln(n+1) - \ln(n+2) - \ln(n+3) - \ln(n+4)$$

step b.) Evaluate $\lim_{n\to\infty} S_n$ to obtain the sum of the series, or state that the series diverges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \underbrace{\ln(24)}_{\text{a number}} - \underbrace{\ln(n+1) - \ln(n+2) - \ln(n+3) - \ln(n+4)}_{\text{go to } -\infty \text{ as } n\to\infty} = -\infty$$

Task: (Copy solution of Textbook Example 8, pg 752) Compute $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$ or state that it diverges.

(Practice)

step a.) Find a formula for the k-th term of the sequence of **partial sums** $\{S_n\}$

step b.) Evaluate $\lim_{n\to\infty} S_n$ to obtain the sum of the series, or state that the series diverges.

The Harmonic Series

Theorem (Textbook Example 9)

The **harmonic series** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$

So it is divergent.

The textbook explains why in Sec 11.2 using partial sums and in Sec 11.3 using improper integrals.

will explain in Sec 11.3

Sec 11.2 (Practice)

If $a_1 + a_2 + a_3 + \dots + a_N = 1 - \frac{1}{N+1}$, evaluate $\sum_{k=1}^{\infty} a_k$.

Determine whether $\sum_{k=1}^{\infty} a_k$ converges.

Write a letter to your future self (a year from now) what it means for an infinite series $a_1 + a_2 + a_3 + \dots$ to be convergent and divergent. Write in complete sentences.

Reference Def 2, pg 748

If $a_1 + a_2 + a_3 + \dots + a_N = -5\left(1 - 2^N\right)$, evaluate $\sum_{k=1}^{\infty} a_k$.

Determine whether $\sum_{k=1}^{\infty} a_k$ converges.

Assuming the pattern continues, determine if $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ is a geometric series. If so, determine its ratio.

Sec 11.2 (Practice) Answer Key

If $a_1 + a_2 + a_3 + \cdots + a_N = 1 - \frac{1}{N+1}$, evaluate $\sum_{k=1}^{\infty} a_k$.
Determine whether $\sum_{k=1}^{\infty} a_k$ converges.

solution: bottom half of Example 8, pg 752

Write a letter to your future self (a year from now) what it means for an infinite series $a_1 + a_2 + a_3 + \cdots$ to be convergent and divergent. Write in complete sentences.

Reference Def 2, pg 748

If $a_1 + a_2 + a_3 + \cdots + a_N = -5\left(1 - 2^N\right)$, evaluate $\sum_{k=1}^{\infty} a_k$.
Determine whether $\sum_{k=1}^{\infty} a_k$ converges.

Answer: $\lim_{N\to\infty} \sum_{k=1}^{N} a_k = \lim_{N\to\infty} \underbrace{-5}_{\text{just a number}} + \overbrace{5 \cdot 2^n}^{\text{goes to } \infty} = \infty$ so $\sum_{k=1}^{\infty} a_k$ diverges

[see also bottom of pg 749 (Example 2)]

Assuming the pattern continues, determine if $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ is a geometric series. If so, determine its ratio.

Sol: Example 3, pg 750

Test for Divergence

Theorem (Textbook Thm 6)

If the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} a_k$ $= 0$.

What does this theorem say? Recall that to any series $\sum a_n$ we associate two sequences:

- the sequence $\{a_k\}$ of its **terms**, and
- the sequence $\{S_n\}$ of its **partial sums**.

The theorem says that if $\sum_{k=1}^{n} a_k$ converges to a number S, then

$$\lim_{n\to\infty} S_n = S \qquad \text{and} \qquad \lim_{k\to\infty} a_k = 0$$

Caution: If the series $\sum_{k=1}^{\infty} a_k$ is divergent, then $\lim_{k\to\infty} a_k$ we cannot say – it depends.

Vocab What is the contrapositive of a statement? Statement: IF P THEN Q

Contrapositive of this statement is "IF (NOT Q) THEN (NOT P)".

E.g. Statement: IF it is snowing, THEN OU is closed

Contrapositive: IF OU is not closed, THEN it is not snowing

The contrapositive is equivalent to the original statement

Test for Divergence (Textbook Thm 7) Contrapositive of Thm 6 above

If $\lim_{k\to\infty} a_k \neq 0$ OR if $\lim_{k\to\infty} a_k$ doesn't exist, then the series $\sum_{k=1}^{\infty} a_k$ is NOT convergent.

divergent

Caution: If $\lim_{k\to\infty} a_k = 0$, then **the test is inconclusive**. We cannot use this test to determine convergence/divergence of $\sum a_k$.

Example: Use the Test for Divergence to determine whether the series $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ diverges, or state that the Test for Divergence is inconclusive.

First step: $$\lim_{k\to\infty} \frac{k}{2k+1} = \lim_{k\to\infty} \frac{\left(\frac{k}{k}\right)}{\left(\frac{2k}{k}\right)+\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{2+\frac{1}{k}} = \frac{1}{2}$$

Second step: The Test for Divergence is conclusive/inconclusive

Since $\lim_{k\to\infty} \frac{k}{2k+1} \neq 0$, the series $\sum \frac{k}{2k+1}$ is divergent

(Practice)

Task: (Copy solution from Example 10, pg 754) Use the Test for Divergence to determine whether the series $\sum_{k=1}^{\infty} \frac{k^2}{5k^2+4}$ diverges, or state that the Test for Divergence is inconclusive.

Do at home

Example: Use the Test for Divergence to determine whether the series $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ diverges, or state that the Test for Divergence is inconclusive.

First step: $\lim_{k\to\infty} \frac{k}{k^2+1} \overset{L'H\ \text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{k\to\infty} \frac{1}{2k} = 0$

Second step: The Test for Divergence is conclusive/inconclusive

Test for Divergence doesn't help. Will learn tools that work in Sec 11.3, 11.4

Properties of Convergent Series

Theorem (Textbook Thm 8)

Suppose c is a number. If $\sum a_k$ and $\sum b_k$ are convergent series, ...

- then the series $\sum c\ a_k$ also **converges** and $\sum c\ a_k =$ $c \sum a_k$
- then the series $\sum a_k + \sum b_k$ also **converges** and $\sum a_k + b_k =$ $\sum a_k + \sum b_k$

Task: (Copy solution of Example 11, pg 755) Evaluate $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$ or state that it diverges.

step a.) First compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ to get 1.

(page 4)

Do at home

step b.) Next, compute $\sum_{n=1}^{\infty} \frac{1}{2^n}$ to get 1.

(page 3)

Do at home

step c.) By Thm 8, we have $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \left(\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \right) + \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) = 3 \cdot 1 + 1.$