Taylor and Maclaurin Series

Suppose for $|x-a|<R$, we have

True for all "nice" functions $f(x)$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots, \quad \Big\} \text{ power series centered at } a$$

then $f(a)=c_0$. We can differentiate both sides with respect to x to get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots,$$

then $f'(a)=c_1$. Again, we have

$$f''(x) = 2c_2 + 3(2)c_3(x-a) + 4(3)c_4(x-a)^2 + 5(4)c_5(x-a)^3 + \ldots,$$

then $f''(a) = 2c_2$. Apply the procedure again to obtain

$$f'''(x) = 3(2)c_3 + 4(3)(2)c_4(x-a) + 5(4)(3)c_5(x-2)^2 + \ldots,$$

then $f'''(a) = 3(2)c_3$. Apply the procedure one more time to obtain

$$f^{(4)}(x) = 4(3)(2)c_4 + 5(4)(3)(2)c_5(x-2) + \ldots,$$

then $f^{(4)}(a) = 4(3)(2)c_4$. By now you can see the pattern. If we continue to differentiate and substitute $x=a$, we obtain

$$f^{(n)}(a) = n(n-1)(n-2)\cdots(2)\,c_n$$
$$= n!\,c_n.$$

So

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}.}$$

Webwork Problem 1

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and $f(0)=14$, $f'(0)=-15$, $f''(0)=-1$, $f'''(0)=-1$,

find the first four terms of $\sum_{n=0}^{\infty} c_n x^n$.

Answer $c_0 = \frac{f(0)}{0!} = 14$ $\quad c_1 = \frac{f'(0)}{1!} = -15$ $\quad c_2 = \frac{f''(0)}{2!} = -\frac{1}{2}$ $\quad c_3 = \frac{f'''(0)}{3!} = -\frac{1}{6}$

$$\boxed{14 - 15x - \frac{1}{2}x^2 - \frac{1}{6}x^3}$$

Theorem
IF f has a power series representation at $x=a$, that is, if

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for} \qquad |x-a|<R,$$

THEN its coefficients are given by

$$c_n=\frac{f^{(n)}(a)}{n!}.$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$=\underbrace{f(a)}_{c_0}+\underbrace{f'(a)}_{c_1}(x-a)+\frac{f''(a)}{\underbrace{2!}_{c_2}}(x-a)^2+\frac{f'''(a)}{\underbrace{3!}_{c_3}}(x-a)^3+\cdots$$

The series is called the **Taylor series** of the function f at $x=a$. For the special case when $a=0$, the Taylor series becomes

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots\cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

Textbook Example 1
Find the Maclaurin series of the function $f(x)=e^x$ and its interval of convergence.

We want to find $\sum_{n=0}^{\infty} c_n x^n$ where $c_n=\frac{f^{(n)}(0)}{n!}$

$f^{(n)}(x)=e^x$ for all n

$f^{(n)}(0)=e^0=1$ for all n, all x

$c_n=\frac{1}{n!}$ for all n, so the Maclaurin series is $\boxed{\sum_{n=0}^{\infty}\frac{1}{n!}x^n}$

Find radius of convergence (using Ratio Test): Let $a_n=\frac{x^n}{n!}$

$$\frac{a_{n+1}}{a_n}=\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}=\frac{x}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=|x|\lim_{n\to\infty}\frac{1}{n+1}=0 \quad \text{for all } x$$

So radius of convergence is ∞.
Interval of convergence is $(-\infty,\infty)$.

Theorem (Textbook Example 2)

The function $f(x) = e^x$ is equal to its Maclaurin series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

Application: Approximate a non-rational number like e (that is, write the first few digits in decimal) on a computer by computing the sum of the first few terms of the series.

$\underbrace{1/e}_{e^{(-1)}}$ is equal to the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

Webwork Problem 10 For the following indefinite integral, find the full power series centered at $x = 0$

$$f(x) = \int \frac{e^{9x} - 1}{2x}\, dx$$

$$f(x) = C + \sum_{n=1}^{\infty} \text{\_\_\_\_\_\_\_\_}$$

$$e^{9x} = \sum_{n=0}^{\infty} \frac{(9x)^n}{n!} = \sum_{n=0}^{\infty} 9^n \frac{1}{n!} x^n$$

$$\frac{e^{9x}}{2x} = \frac{1}{2} \sum_{n=0}^{\infty} 9^n \frac{1}{n!} x^{n-1} = \frac{1}{2}\left[9^0 \frac{1}{0!} \frac{1}{x} + \sum_{n=1}^{\infty} 9^n \frac{1}{n!} x^{n-1} \right]$$

$$\int \frac{e^{9x}}{2x}\, dx = \frac{1}{2}\left[\ln x + \sum_{n=1}^{\infty} 9^n \frac{1}{n!} \frac{x^n}{n} \right] + C$$

$$\int \frac{1}{2x}\, dx = \frac{1}{2} \ln x + D$$

So the final answer is $\boxed{\sum_{n=1}^{\infty} \frac{9^n}{2} \frac{1}{n!} \frac{1}{n} x^n + C}$

Example Compute the Maclaurin seris for $\cos x$.

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$\vdots$

The pattern repeats in a cycle of four

$f^{(4n)}(0) = 1$

$f^{(4n+1)}(0) = 0$

$f^{(4n+2)}(0) = -1$

$f^{(4n+3)}(0) = 0$

The Maclaurin series is $1x^0 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$

(Reindex so that $n=0$ $n=1$ $n=2$ $n=3$ $n=4$)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Fact: The Maclaurin series for $\cos x$ is equal to $\cos x$.

Copy Example 4, pg 804
Copy the computation for the Maclaurin series for $\sin x$ (you can skip the proof that the Maclaurin series is equal to $\sin x$).

Do after class

Webwork Problem 2 (i) Find the Taylor series for $f(x) = \ln x$ centered at 10.
(ii) Find the interval of convergence.

Answer: The Taylor series for $\ln x$ centered at 10 is $\displaystyle\sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(10)}{n!}}_{c_n}(x-10)^n$

$f(x) = \ln x$ $\qquad f(10) = \ln 10$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = -\frac{3.2}{x^4}$

$f^{(5)}(x) = \frac{4.3.2}{x^5}$

$f^{(6)}(x) = -\frac{5.4.3.2}{x^6}$

$f^{(7)}(x) = \frac{6.5.4.3.2}{x^7}$

Practice

Pattern for $n \geq 1$:

$$f^{(n)}(x) = \frac{(n-1)!\,(-1)^{n-1}}{x^n}$$

so $c_n = \frac{f^{(n)}(10)}{n!} = \frac{(n-1)!\,(-1)^{n-1}}{10^n}\,\frac{1}{n!} = \frac{(-1)^{n-1}}{10^n\, n}$

$$\sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(10)}{n!}}_{c_n}(x-10)^n = \boxed{\ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{10^n\, n}(x-10)^n}$$

(ii) Find radius of convergence (using Ratio Test): Let $a_n = \frac{(-1)^{n-1}}{10^n\, n}(x-10)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-10)^{n+1}}{10^{n+1}(n+1)} \cdot \frac{10^n\, n}{(x-10)^n}\right| = \left|\frac{(x-10)}{10}\,\frac{n}{n+1}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x-10}{10}\,\frac{n}{n+1}\right| = \left|\frac{x-10}{10}\right| \lim_{n\to\infty}\frac{n}{n+1} = \left|\frac{x-10}{10}\right|$$

By Ratio Test, $\sum_{n=1}^{\infty}|a_n|$ converges when $\left|\frac{x-10}{10}\right| < 1$

$\iff |x-10| < 10$ This is the radius of convergence

$-10 < x-10 < 10$

$10-10 < x < 10+10$

Check $x = 10-10 = 0$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{10^n\, n}(-10)^n = \sum_{n=0}^{\infty}(-1)^{n-1}(-1)^n\frac{10^n}{10^n}\frac{1}{n} = \sum_{n=0}^{\infty}-\frac{1}{n}$ is the Harmonic series (divergent)

Check $x = 10+10 = 20$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(10)^n}{10^n\, n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is convergent by Alternating Series Test

($\{\frac{1}{n}\}$ is a decreasing sequence $\frac{1}{n} > \frac{1}{n+1}$ for all $n \geq 0$)

Interval of Convergence: $\boxed{(0, 20]}$

<u>Commonly used Maclaurin Series</u>, see Table 1 page 808.

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$.

- $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$ for $-1 < x < 1$.

- $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $-1 \le x < 1$.
  - $\sum_{n=1}^{\infty} \frac{1}{n\, 2^n} = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n} = -\ln\left(1-\frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \ln(2)$

- $\ln(1+x) = \ln(1-(-x)) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ for $-1 < x \le 1$.

- arctan x $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $-1 \le x \le 1$.
  - $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}$.

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $-\infty < x < \infty$.
  - $\sum_{n=0}^{\infty} \frac{1}{n!} = e^1$.

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for $-\infty < x < \infty$.
  - $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin(\pi) = 0$.

- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all x.