Sec 11.1 Sequences

A <u>sequence</u> is an ordered collection of objects

<u>Examples</u>

* A sequence of letters

* △ ■ ○ ▲ □ ● △ _ _ _ ...

In calculus, a <u>sequence</u> is a list of numbers

indexed by the natural numbers $1, 2, 3, 4, \ldots$

Notation: $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

The index doesn't have to start at 1, <u>ex</u> $\{a_n\}_{n=0}^{\infty}$ or $\{a_n\}_{n=5}^{\infty}$

<u>Examples of sequences</u>

* $1, 3, 5, 7, 9, \ldots$ is the sequence of odd natural numbers

  formula $a_n = 2n - 1$ for $n = 1, 2, 3, \ldots$

* $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2n^2 - 3n + 1$.

  Write the first three terms of $\{a_n\}_{n=1}^{\infty}$

$$a_1 = 2(1)^2 - 3(1) + 1 = 0$$

$$a_2 = 2(4) - 3(2) + 1 = 3$$

$$a_3 = 2(9) - 3(3) + 1 = 10$$

* Find a formula for the general term a_n for the sequence $\{1, -3, 5, -7, 9, \dots\}$:

If starting index is $n=1$: $a_n = (2n-1)(-1)^{n+1}$ for $n=1,2,3,\dots$

or $a_n = -(2n-1)(-1)^n$

If starting index is $n=0$: $a_n = (2n+1)(-1)^n$ for $n=0,1,2,\dots$

* Find a formula for the general term a_n of the sequence $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots\right\}$:

$$\underset{a_1}{\frac{3}{5}}, \quad \underset{a_2}{-\frac{4}{25}}, \quad \underset{a_3}{\frac{5}{125}}, \quad \underset{a_4}{-\frac{6}{625}}, \quad \underset{a_5}{\frac{7}{3125}}$$

- If starting index is $n=1$:
- The signs alternate positive & negative, so we need to multiply by $(-1)^{(\text{something})}$. a_1 is positive, so multiply by $(-1)^{n+1}$ or $(-1)^{n-1}$.
- Numerators are $\underset{a_1}{3}, \underset{a_2}{4}, \underset{a_3}{5}, \underset{a_4}{6}, \underset{a_5}{7}, \dots$: $(n+2)$ in general
- Denominators are $5, 25, 125, 625, 3125$: 5^n in general

$$\underset{a_1}{5^1} \quad \underset{a_2}{5^2} \quad \underset{a_3}{5^3} \quad \underset{a_4}{5^4} \quad \underset{a_5}{5^5}$$

- $a_n = (-1)^{n+1} \dfrac{n+2}{5^n}$ for $n=1,2,3,\dots$
- If starting index is $n=0$: $a_n = (-1)^n \dfrac{n+3}{5^{n+1}}$ or $\dfrac{(-1)^n}{5} \dfrac{n+3}{5^n}$ for $n=0,1,2,\dots$

* The Fibonacci sequence is defined recursively by

$$a_1 = 1, \quad a_2 = 1, \quad \underbrace{a_{n+2} = a_n + a_{n+1}}_{\text{each term is the sum of the previous two terms}} \quad \text{for } n = 1, 2, 3, \ldots$$

First few terms of the Fibonacci sequence:

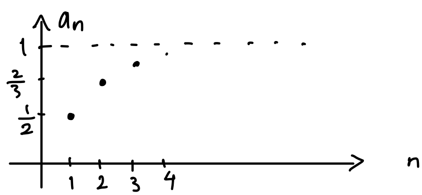
$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

* The sequence $a_n = \dfrac{n}{n+1}$ for $n = 1, 2, 3, \ldots$

Table:

n	1	2	3	4	...	n
a_n	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	...	$\frac{n}{n+1}$

Graph:



(The terms of $a_n = \dfrac{n}{n+1}$ seem to approach 1 as n gets large.)

The difference $1 - a_n = 1 - \dfrac{n}{n+1}$

$$= \frac{n+1-n}{n+1}$$

$$= \frac{1}{n+1}$$

can be made as small as we like by taking large enough n.

The notation for this is $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$.

In general, writing $\boxed{\displaystyle\lim_{n\to\infty} a_n = L}$ means:

the terms of the sequence $\{a_n\}$ approach L as n becomes large.

New vocab (memorize)

* A sequence $\{a_n\}$ has limit L (a number) &

we write $\displaystyle\lim_{n\to\infty} a_n = L$ or write $a_n \to L$ as $n \to \infty$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

* If $\underbrace{\displaystyle\lim_{n\to\infty} a_n}_{\text{(is a number)}}$ exists, say $\{a_n\}$ converges (or is convergent).

* otherwise, say $\{a_n\}$ diverges (or is divergent or is not convergent).

Ex Is the sequence $a_n = \dfrac{2n}{n+1}$ convergent or divergent?

$$\lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2n}{n+1}\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{2}{\frac{n}{n}+\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}}$$

$$= \frac{\left(\displaystyle\lim_{n\to\infty} 2\right)}{\left(\displaystyle\lim_{n\to\infty} 1\right)+\left(\displaystyle\lim_{n\to\infty} \frac{1}{n}\right)}$$

$$= \frac{2}{1+0} = 2$$

(a number)

* We say: $\{a_n\}$ has limit 2.

* Since $\displaystyle\lim_{n\to\infty} \frac{2n}{n+1}$ exists, we say $\{a_n\}$ is convergent.

New vocab

Writing $\lim_{n\to\infty} a_n = \infty$ means:

for $\underbrace{\text{every positive number } M}_{\text{no matter how big}}$,

there is an integer N such that

if $n > N$ then $a_n > M$.

Say $\{a_n\}$ diverges to ∞.

$\lim_{n\to\infty} a_n = -\infty$ means:

for every positive number M,

there is an integer N

such that

if $n > N$ then $a_n < -M$.

Say $\{a_n\}$ diverges to $-\infty$.

Ex Is $a_n = \dfrac{-n}{\sqrt{10+n}}$ convergent?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-n}{\sqrt{10+n}} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{-1}{\sqrt{\frac{10}{n^2}+\frac{n}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{-1}{\sqrt{\frac{10}{n^2}+\frac{1}{n}}}$$

numerator $= -1 \to -1$ as $n\to\infty$

denominator $= \sqrt{\frac{10}{n^2}+\frac{1}{n}} \to 0$ as $n\to\infty$

$$= -\infty$$

* $\lim_{n\to\infty} a_n$ does not exist, so $\{a_n\}$ diverges (is not convergent).

* $\lim_{n\to\infty} a_n = -\infty$ means $\{a_n\}$ diverges in a special way:

Say $\{a_n\}$ diverges to $-\infty$.

Thm Let f be any function.

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer,

then $\lim_{n\to\infty} a_n = L$ (Upshot: We can replace x with n)

Ex (Application of Thm) Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$:

Let $f(x) = \frac{\ln x}{x}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \underset{\uparrow}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} \overset{(*)}{=} 0$$

l'Hospital's Rule
type "$\frac{\infty}{\infty}$"

Since $f(n) = a_n$ for $n = 1, 2, 3, \ldots,$ we can apply above Thm:

$$\lim_{n\to\infty} a_n \underset{\text{Thm}}{=} \lim_{x\to\infty} f(x) \underset{\text{by } (*)}{=} \boxed{0.}$$

Thm (Limit laws for convergent sequences)

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a number,

then
* $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
* $\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n$
* $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

Squeeze Thm

If * $a_n \le b_n \le c_n$ for $n \geqslant N$, AND

* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

THEN $\lim_{n\to\infty} b_n = L$

(If b_n is bounded above & below by two sequences converging to L, then b_n converges to L.)

Thm (special case of squeeze Thm)

If $\lim_{n\to\infty} |b_n| = 0$ then $\lim_{n\to\infty} b_n = 0$.

Ex (of squeeze Thm)

Is $b_n = \frac{(-1)^n}{n}$ convergent ?

$$\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

By Squeeze Thm, $\lim_{n\to\infty} b_n = 0$. So $\lim_{n\to\infty} b_n$ exists.

So $\{b_n\}$ is convergent.

Thm If $\lim_{n\to\infty} a_n = L$ and function f is continuous at L,

then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

Upshot: can bring $\lim_{n\to\infty}$ inside brackets if

f is continuous at L.

Ex (of thm)

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = ?$$

Let $a_n := \frac{\pi}{n}$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\pi}{n} = 0 \overset{=:}{} L$.

Let $f(x) := \sin x$. Then $f(x)$ is continuous at $0 \overset{=}{} L$.

So $\lim_{n\to\infty} f(a_n) \underset{\text{by Thm}}{=} f\left(\lim_{n\to\infty} a_n\right) = f(0)$

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) \underset{\text{by Thm}}{=} \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin(0) = \boxed{0}.$$

New vocab (memorize)

$a_n = r^n$ (like $a_n = (\frac{1}{2})^n$, $a_n = (-2)^n$, $a_n = 1^n$, $a_n = (-1)^n$)

"ratio"

is called a <u>geometric sequence</u>.

<u>Ex</u>

* $\lim_{n\to\infty} (\frac{1}{2})^n = 0$, say $\{\frac{1}{2^n}\}$ <u>converges to 0</u>

* $\lim_{n\to\infty} 1^n = 1$, say $\{1\}$ <u>converges to 1</u>

* $\lim_{n\to\infty} 2^n = \infty$, say $\{2^n\}$ <u>diverges to ∞</u>

not a number

does not exist

* $\lim_{n\to\infty} (-\frac{2}{3})^n = 0$ by squeeze Thm.

Say $\{(-\frac{2}{3})^n\}$ converges to 0.

* $\lim_{n\to\infty} (-1)^n$ doesn't exist. Say $\{(-1)^n\}$ diverges.

* $\lim_{n\to\infty} (-\frac{3}{2})^n$ doesn't exist. Say $\{(-\frac{3}{2})^n\}$ diverges.

<u>Fact</u> The geometric sequence $\{r^n\}$

is convergent if $-1 < r \leq 1$: (like $r = 1, \frac{1}{2}, -\frac{2}{3}$)

$\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$

$\lim_{n\to\infty} 1^n = 1$

$\{r^n\}$ diverges if $r \leq -1$ or $1 < r$ (like $r = -1, \frac{-3}{2}, 2$)

New vocab

* $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$:

$$a_1 < a_2 < a_3 < \cdots$$

* $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$:

$$a_1 > a_2 > a_3 > \cdots$$

* $\{a_n\}$ is monotonic if it is either increasing or decreasing.

Ex Is $\frac{3}{n+5}$ monotonic?

$$a_1 = \frac{3}{6} > a_2 = \frac{3}{7} > a_3 = \frac{3}{8} > \cdots$$

$$a_n = \frac{3}{n+5} > a_{n+1} = \frac{3}{n+6} \quad \text{for all } n = 1, 2, \ldots$$

So $\{a_n\}$ is decreasing, so $\{a_n\}$ is monotonic.

New vocab

* $\{a_n\}$ is bounded above if there is a number M such that $a_n \leq M$ for all $n \geq 1$.

* $\{a_n\}$ is bounded below if there is a number m such that $m \leq a_n$ for all $n \geq 1$.

* Say $\{a_n\}$ is bounded if $\{a_n\}$ is bounded above and below.

Ex $\frac{3}{n+5}$ lower bounds: $0, -\frac{1}{2}$ upper bounds: $\frac{3}{6}, 1, 1000$

Monotonic Sequence Thm

If $\{a_n\}$ is bounded and monotonic, then $\{a_n\}$ converges.

Ex $\left\{\frac{3}{n+5}\right\}$ is decreasing and bounded,

so by the monotonic sequence thm,

$\left\{\frac{3}{n+5}\right\}$ converges.

True or false?

1. If a sequence $\{a_n\}$ is bounded, then $\{a_n\}$ is convergent.

False. Counterexample:

$\{1, -1, 1, -1, \ldots\}$ is bounded by -1 and 1

but it diverges.

2. If $\{a_n\}$ is monotonic, then $\{a_n\}$ is convergent.

False. Counterexample:

Let $a_n = n$. Then $\{a_n\}$ is increasing

but $\lim_{n\to\infty} a_n = \infty$

so $\{a_n\}$ is divergent.

3. If $\{a_n\}$ is convergent, then $\{a_n\}$ is monotonic.

False. Counterexample: $a_n = \frac{(-1)^n}{n}$ is convergent but not monotonic (neither increasing nor decreasing).