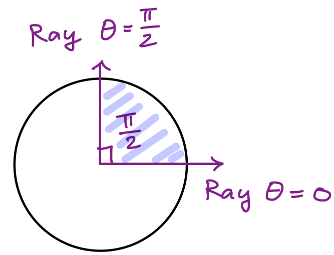
# Sec 10.4 Part I: Areas in polar coordinates

Area of the sector of a circle swept out by

a $\frac{\pi}{2}$ angle with radius $r$ is $\frac{1}{4} r^2 \pi = \frac{1}{2} r^2 \left(\frac{\pi}{2}\right)$

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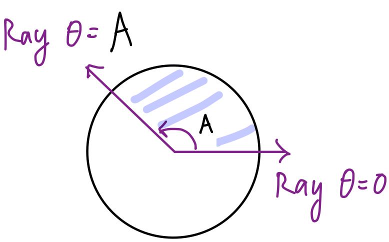
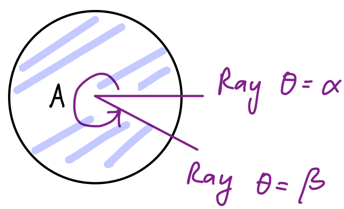
Area of the sector of a circle swept out by

an angle $A$ with radius $r$ is $\frac{1}{2} r^2 A$ ← Here I replace $\frac{\pi}{2}$ with $A$




$A = \beta - \alpha$

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The area of the region bounded by the graph $r = f(\theta)$

between two rays $\theta = \alpha$ and $\theta = \beta$ is

$$\boxed{\int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta}$$

or think $\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$

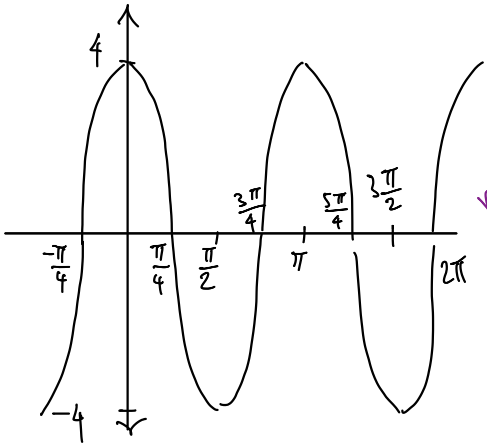
Make sure $\alpha < \beta$

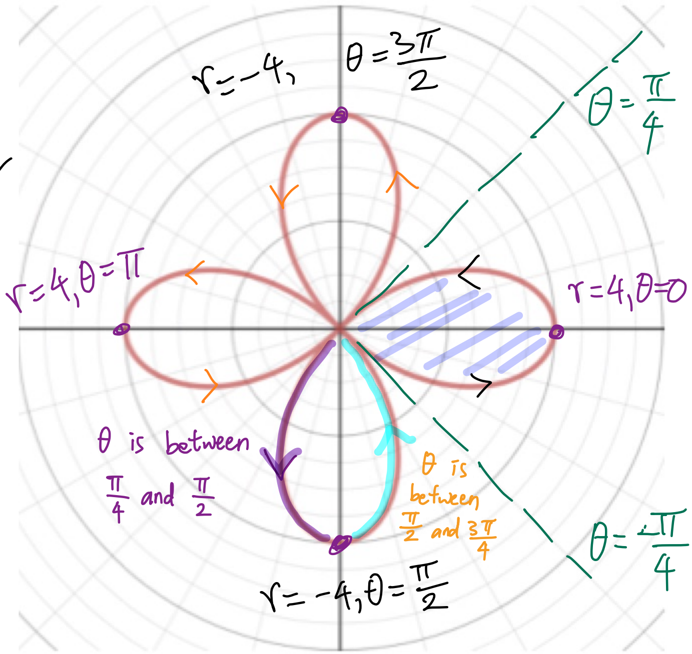
Example: #1

Find the area enclosed by one loop of the four-leaved rose $r = 4\cos 2\theta$. Perform a reality check against your result.

Step 1: Sketch the polar curve

$r(\theta) = 4\cos(2\theta)$




Step 2:

Apply area formula

$$\text{Area} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}[r(\theta)]^2\, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}[4\cos 2\theta]^2\, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} 4^2 [\cos(2\theta)]^2\, d\theta$$

(Sec 7.2)

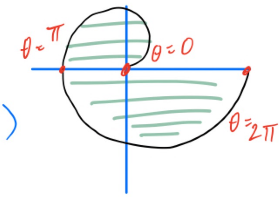
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} 4^2 \frac{1+\cos(4\theta)}{2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\,(1+\cos(4\theta))\, d\theta$$

$$= 4\left[\theta + \frac{\sin(4\theta)}{4}\right]\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 4\left[\left[\frac{\pi}{4} + \frac{\sin\pi}{4}\right] - \left[-\frac{\pi}{4} + \frac{\sin(-\pi)}{4}\right]\right]$$

$$= \boxed{2\pi}$$ (about 6, looks reasonable)

## Example #2

Find the area of the shaded region enclosed by the polar curve $r = \sqrt{\theta}$.




Answer:

The curve is an infinite spiral for $\theta$ in $[0, \infty)$ but the shaded region is enclosed by the curve $r = \sqrt{\theta}$ from $0$ to $2\pi$ (and the positive polar axis).

So the area of the shaded region is

$$\int_0^{2\pi} \frac{1}{2}\left(\sqrt{\theta}\right)^2 d\theta = \int_0^{2\pi} \frac{1}{2}\theta \, d\theta$$

$$= \frac{1}{2}\frac{\theta^2}{2}\Bigg|_0^{2\pi}$$

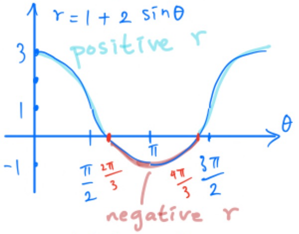
$$= \frac{(2\pi)^2}{4} - 0$$

$$= \boxed{\pi^2}$$

## Example #3

Find the area of the region inside the larger loop and outside the smaller loop of the polar curve $r = 1 + 2\cos\theta$.

① Sketch curve

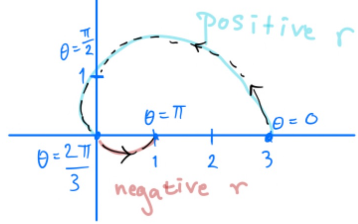



$0 = 1 + 2\cos\theta$

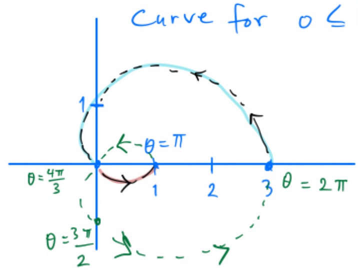
$-\frac{1}{2} = \cos\theta \Rightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

Curve for $0 \leq \theta \leq \pi$:

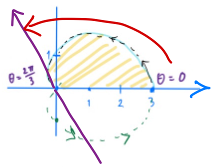



Curve for $0 \leq \theta \leq 2\pi$:




② Compute area inside the larger loop $A_{Larger}$

$A_{larger}$ is twice the area of the shaded region




$$A_{Larger} = 2\int_0^{\frac{2\pi}{3}} \frac{1}{2}\left(1 + 2\cos\theta\right)^2 \, d\theta$$

$$= \int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + 4\cos^2\theta \, d\theta$$

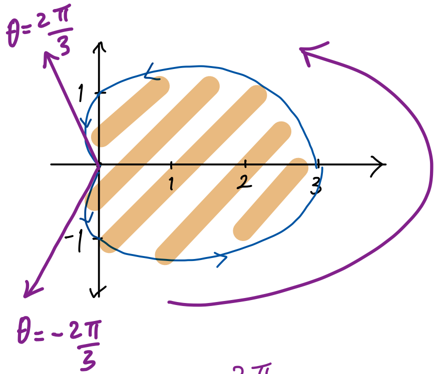
$$= \int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + \frac{4}{2}\left(1 + \cos(2\theta)\right) \, d\theta$$

$$= \boxed{2\pi + \frac{3}{2}\sqrt{3}}$$

(Con't ⟶)

con't with (2)

Alternatively, we can compute area inside the larger loop by bounding the curve between rays $\theta = -\frac{2\pi}{3}$ and $\theta = \frac{2\pi}{3}$




$$A_{\text{larger}} = \int_{\theta = -\frac{2\pi}{3}}^{\theta = \frac{2\pi}{3}} \frac{1}{2}\left(1 + 2\cos\theta\right)^2 \, d\theta$$

$$= \cdots$$

[use trig identity $(\cos x)^2 = \frac{1 + \cos(2x)}{2}$]

$$= \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \frac{3}{2} + 2\cos\theta + \cos(2\theta) \, d\theta$$

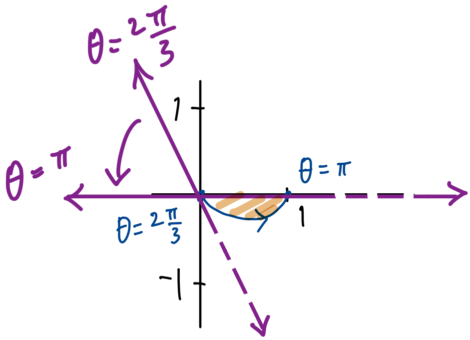
$$= \frac{3}{2}\theta + 2\sin\theta + \frac{\sin(2\theta)}{2} \Bigg|_{\theta = -\frac{2\pi}{3}}^{\theta = \frac{2\pi}{3}}$$

$$= \left[\frac{3}{2}\,\frac{2\pi}{3} + 2\sin\frac{2\pi}{3} + \frac{\sin\left(\frac{4\pi}{3}\right)}{2}\right] - \left[\frac{3}{2}\left(-\frac{2\pi}{3}\right) + 2\sin\left(-\frac{2\pi}{3}\right) + \frac{\sin\left(-\frac{4\pi}{3}\right)}{2}\right]$$

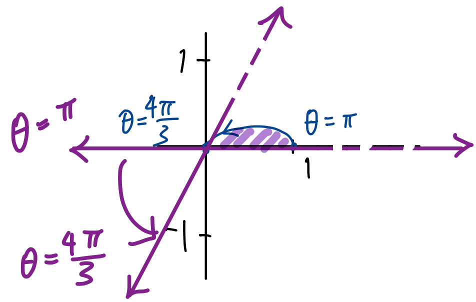
$$= 2\pi + 4\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$= \boxed{2\pi + \frac{3}{2}\sqrt{3}}$$

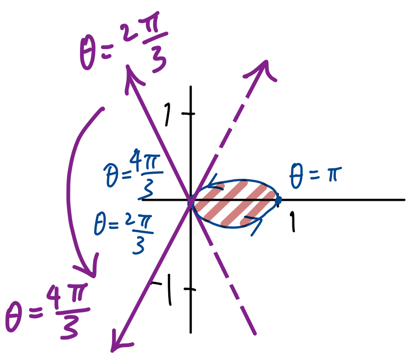
③ Compute area inside the smaller loop




$$\int_{\theta=\frac{2\pi}{3}}^{\theta=\pi} \frac{1}{2}(1+2\cos\theta)^2\, d\theta = \frac{\pi}{2} - \frac{3}{4}\sqrt{3}$$




$$\int_{\theta=\pi}^{\theta=\frac{4\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2\, d\theta = \frac{\pi}{2} - \frac{3}{4}\sqrt{3}$$




Area inside the smaller loop is

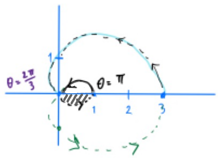
$$A_{\text{smaller}} = \int_{\theta=\frac{2\pi}{3}}^{\theta=\frac{4\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2\, d\theta = \boxed{\pi - \frac{3}{2}\sqrt{3}}$$

(con't →)

(con't with 3)

3 Compute area inside the smaller loop $A_{\text{Smaller}}$

$A_{\text{smaller}}$ is twice the area of the shaded region

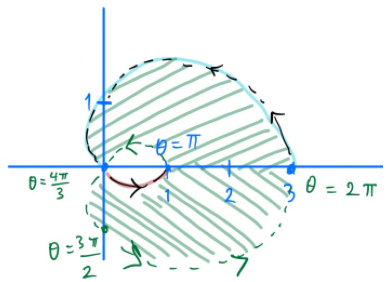



$$A_{\text{smaller}} = 2 \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}(1 + 2\cos\theta)^2 \, d\theta$$

$$= \boxed{\pi - \frac{3}{2}\sqrt{3}}$$

Alternative method

4 Compute the area of the region inside the larger loop and outside the smaller loop.




The desired area is the area from step 2 minus the area from step 3.

$$A = A_{\text{Larger}} - A_{\text{smaller}}$$

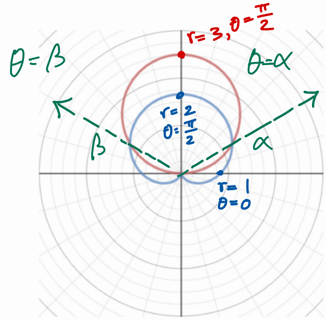
$$= \left(2\pi + \frac{3}{2}\sqrt{3}\right) - \left(\pi - \frac{3}{2}\sqrt{3}\right)$$

$$= \pi + 3\sqrt{3}$$

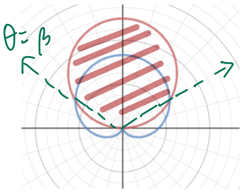
Example: #4

Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$. Perform a reality check against your result.

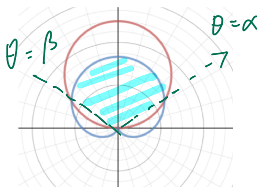
Draw unit circle w/ $\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$ and labels




Inside circle




inside cardioid




Find $\alpha$ and $\beta$: Set circle = cardioid

(between 0 and $2\pi$)

$$3\sin\theta = 1 + \sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

So $\alpha = \frac{\pi}{6}$, $\beta = \frac{5\pi}{6}$

$$\text{Area} = \int_{\theta=\pi/6}^{\theta=5\pi/6} \frac{1}{2}(r_{\text{circle}})^2 \, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(r_{\text{cardioid}})^2 \, d\theta$$

$$A = \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 \, d\theta - \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (1 + \sin\theta)^2 \, d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$A = 2\left[\tfrac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta \, d\theta - \tfrac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta) \, d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta) \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta) \, d\theta \qquad \left[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)\right]$$

$$= 3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2} = \pi$$

# Sec 10.4 Part II: Lengths in polar coordinates

Goal:
Find the length of a polar curve $r = f(\theta)$, $a \le \theta \le b$.

Write the parametric equations of the curve with $\theta$ as parameter:

Conversion formula from 10.3

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

Arc length formula from 10.2:

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \quad \text{if } t \text{ is the parameter}$$

$$\int_{\theta=a}^{\theta=b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta \quad \text{if } \theta \text{ is the parameter}$$

- $\frac{dx}{d\theta} = \frac{d}{d\theta}\left[f(\theta)\cos\theta\right] = \frac{df}{d\theta}\cos\theta + f(\theta)(-\sin\theta) = \frac{df}{d\theta}\cos\theta - \sin\theta\, f(\theta)$
- $\frac{dy}{d\theta} = \frac{d}{d\theta}\left[f(\theta)\sin\theta\right] = \frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{df}{d\theta}\right)^2(\cos\theta)^2 - 2\frac{df}{d\theta}\cos\theta\sin\theta f(\theta) + (f(\theta))^2(\sin\theta)^2$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(\frac{df}{d\theta}\right)^2(\sin\theta)^2 + 2\frac{df}{d\theta}\sin\theta f(\theta)\cos\theta + (f(\theta))^2(\cos\theta)^2$$

+

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{df}{d\theta}\right)^2\Big[\underbrace{(\cos\theta)^2 + (\sin\theta)^2}_{1}\Big] + [f(\theta)]^2\Big[\underbrace{(\sin\theta)^2 + (\cos\theta)^2}_{1}\Big]$$

$$= \left(\frac{df}{d\theta}\right)^2 + (f(\theta))^2$$

(Con't →)

So $\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} = \sqrt{\left(\frac{df}{d\theta}\right)^2+\left(f(\theta)\right)^2}$ $\Bigg($ or $\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}$ for short

since the curve is defined

by $r=f(\theta)\Bigg)$

So $\boxed{\text{Arc length is } \int_{\theta=a}^{\theta=b} \sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\, d\theta}$

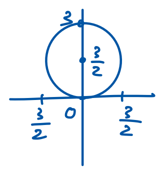
Recall: $r = 3\sin(\theta)$ is a circle:

$r = 3\,\frac{y}{r} \Rightarrow r^2 = 3y$

$\Rightarrow x^2+y^2 = 3y$

$\Rightarrow x^2+y^2-3y+1.5^2 = 1.5^2$

$\Rightarrow x^2+\left(y-\frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$




<u>Webwork</u>

Find the exact length of the polar curve

$$r = 3\sin(\theta),\ \ 0 \le \theta \le \pi/3.$$

$\frac{dr}{d\theta} = 3\cos\theta$ , $\left(\frac{dr}{d\theta}\right)^2 = 3^2(\cos\theta)^2$

$r = 3\sin\theta$ , $r^2 = 3^2(\sin\theta)^2$ +

$$\left(\frac{dr}{d\theta}\right)^2+r^2 = 3^2\Big[\underbrace{(\cos\theta)^2+(\sin\theta)^2}_{1}\Big] = 3^2$$

$$\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2} = \sqrt{3^2} = 3$$

$$\int_{\theta=0}^{\theta=\frac{\pi}{3}} \sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\, d\theta = \int_0^{\frac{\pi}{3}} 3\, d\theta = 3\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{3}} = 3\,\frac{\pi}{3} = \boxed{\pi}$$