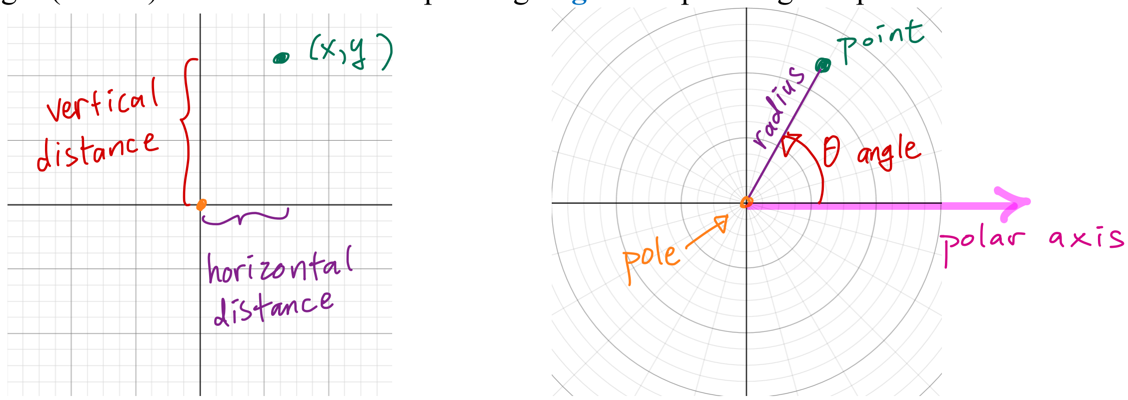
Polar Coordinates

(In Multivariable Calculus: polar coordinates → cylindrical and spherical coordinates in 3D)

Instead of using **horizontal distance** and **vertical distance** from the axes, we use the distance from the origin (**radius**) as well as the corresponding **angle** to express a given point.

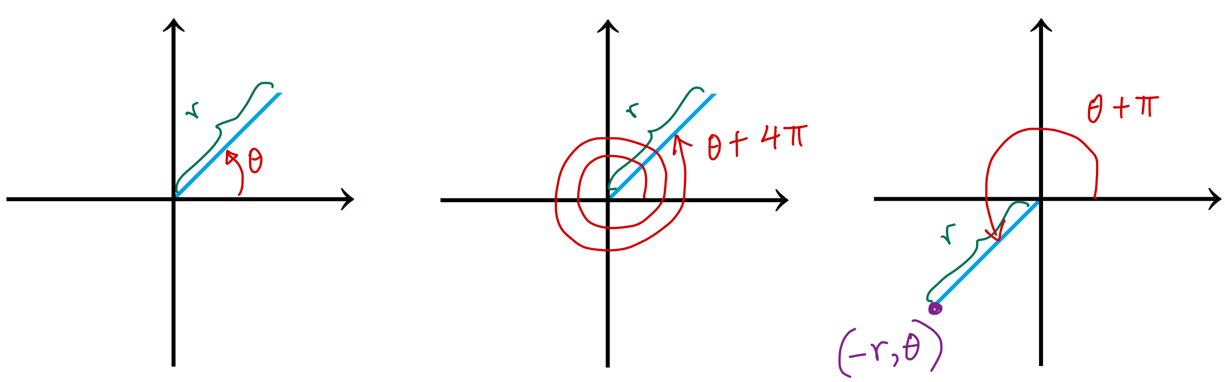


- The origin is called the **pole**.
- The positive x-axis is called the **polar axis**.
- The polar coordinates for a point P have the form (r,θ), where …
  - r is the distance from the origin to P, and
  - θ is an angle starting from the $\underbrace{\text{positive } x\text{-axis}}_{\text{polar axis}}$ and ending on the ray that passes through the pole and the point P.

- Positive angles are measured **counterclockwise** from the positive x-axis.

Caution (Webwork Problem 1)
The representation for a given point is **not unique** in the polar coordinate system.

(r,θ), $\underbrace{(r, \theta + 2n\pi)}_{\text{any even multiple of } \pi}$ and $\underbrace{(-r, \theta + (2n+1)\pi)}_{\text{changing angle by } 180^\circ}$ refer to the same point.



The origin is specified as $(0,\theta)$ in polar coordinates, where θ is any angle.

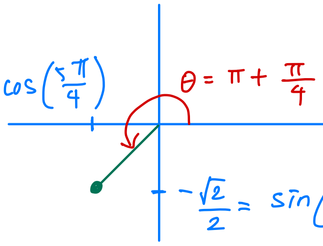
$\cos(\pi+\theta) = -\cos(\theta)$
$\sin(\pi+\theta) = -\sin(\theta)$

$5\frac{\pi}{4} = \pi + \frac{\pi}{4}$

Find other representations for the same point (Webwork Problem 1):

Graph the point $\left(1, \frac{5\pi}{4}\right)$ in polar coordinates. Give ~~two~~ four alternative representations for the point.

$-\frac{\sqrt{2}}{2} = \cos\left(\frac{5\pi}{4}\right)$



$-\frac{\sqrt{2}}{2} = \sin\left(\frac{5\pi}{4}\right)$

$\left(1, \frac{5\pi}{4} + 2\pi\right), \left(1, \frac{5\pi}{4} - 2\pi\right)$

$\left(-1, \frac{5\pi}{4} + \pi\right), \left(-1, \frac{5\pi}{4} - 3\pi\right)$

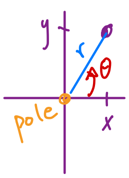
Converting Between Cartesian and Polar Coordinates

Procedure **Converting Coordinates**

A point with polar coordinates (r,θ) has Cartesian coordinates (x, y), where

$x = \left(\text{horizontal distance}\right) = r\cos\theta$, $y = \left(\text{vertical distance}\right) = r\sin\theta$.

A point with Cartesian coordinates (x, y) has polar coordinates (r,θ), where



$r = \left(\text{distance from pole}\right) = \sqrt{x^2+y^2}$, $\theta = \left(\text{angle from polar axis}\right) = \arctan\left(\frac{y}{x}\right)$.

Polar to Cartesian (Webwork Problem 2):

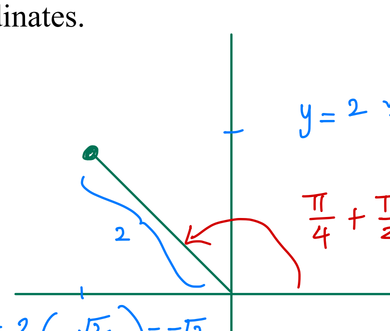
Express the point with polar coordinates $\left(\overset{r}{2}, \overset{\theta}{\frac{3\pi}{4}}\right)$ in Cartesian coordinates.

$x = r\cos\theta = 2\cos\frac{3\pi}{4} = -\sqrt{2}$

$y = r\sin\theta = 2\sin\frac{3\pi}{4} = \sqrt{2}$

Answer: $\left(-\sqrt{2}, \sqrt{2}\right)$

$y = 2\frac{\sqrt{2}}{2} = \sqrt{2}$



$x = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

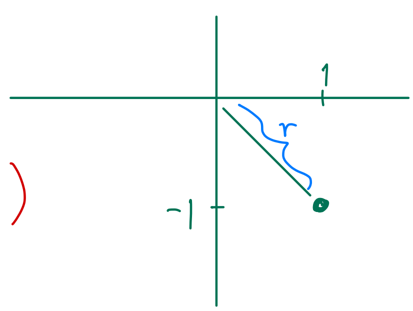
Cartesian to polar (Webwork Problem 4):

Express the point with Cartesian coordinates $(1,-1)$ in polar coordinates.

$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = -\frac{\pi}{4}$ or $-\frac{\pi}{4} + 2n\pi$

(from picture or compute $\theta = \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$)

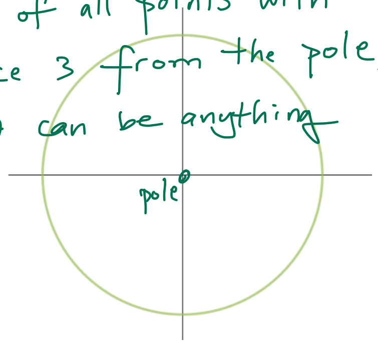


Converting Between Cartesian and Polar Equations

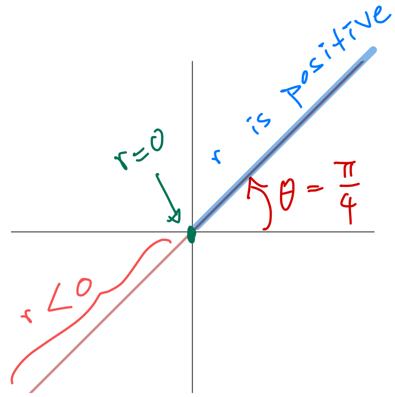
A curve in polar coordinates is the set of points that satisfy an equation in r and θ. Some sets of points are easier to describe in polar coordinates than in Cartesian coordinates.

Examples:

The set of all points with distance 3 from the pole, angle θ can be anything

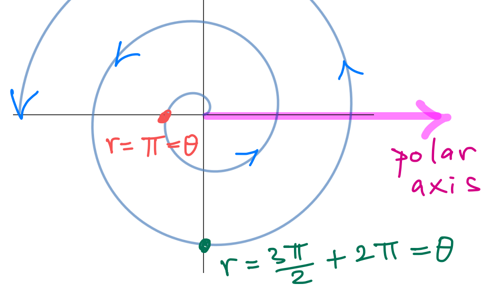


polar equation $r = 3$



polar equation $\theta = \dfrac{\pi}{4}$

All points where (distance from pole) = (angle from polar axis)



polar equation $r = \theta$

(Graph with Desmos: https://www.desmos.com/calculator/j6ha36k9zi)

Polar to Cartesian (Webwork Problem 10):
Convert the polar equation $r = 8\sin\theta$ to a Cartesian equation.

page 2: $x = r\cos\theta$, $y = r\sin\theta \iff \frac{y}{r} = \sin\theta$

Plug in $\frac{y}{r} = \sin\theta$ into $r = 8\sin\theta$: $r = 8\frac{y}{r} \iff r^2 = 8y$

page 2: $r^2 = x^2 + y^2$

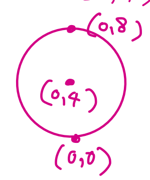
Plug in $r^2 = x^2 + y^2$ into $r^2 = 8y$: $\boxed{x^2 + y^2 = 8y}$

$x^2 + y^2 - 8y = 0$

$x^2 + y^2 - 8y + 4^2 = 4^2$

$x^2 + (y-4)^2 = 4^2$

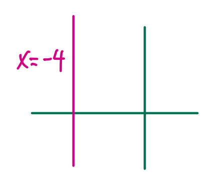
circle with radius 4, centered at (0,4)



Polar to Cartesian (Webwork Problem 13):
Convert the polar equation $r\cos\theta = -4$ to a Cartesian equation.

page 2: $x = r\cos\theta$

Plug in $x = r\cos\theta$ into $r\cos\theta = -4$: $\boxed{x = -4}$



Cartesian to polar equation (Webwork Problem 9):
Convert the Cartesian equation $x^2(x^2 + y^2) = 4y^2$ to a polar equation.

page 2: $r^2 = x^2 + y^2$

Plug into $x^2(x^2 + y^2) = 4y^2$: $x^2 r^2 = 4y^2$

page 2: $x = r\cos\theta$, $y = r\sin\theta$

Plug into $x^2 r^2 = 4y^2$: $r^2(\cos\theta)^2 r^2 = 4r^2(\sin\theta)^2$

$r^2 = 4(\tan\theta)^2$

$\boxed{r = 2\tan\theta}$

<u>Graphing in Polar Coordinates</u>

Sketch the polar equation $r = 1 + \sin\theta$.

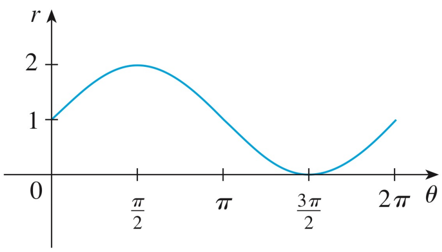
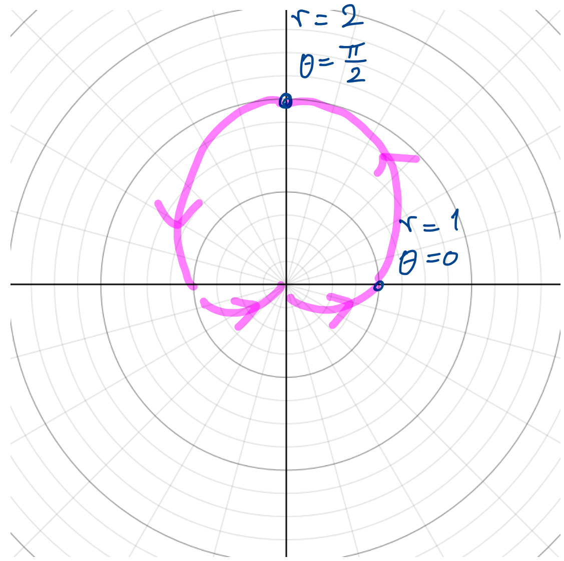
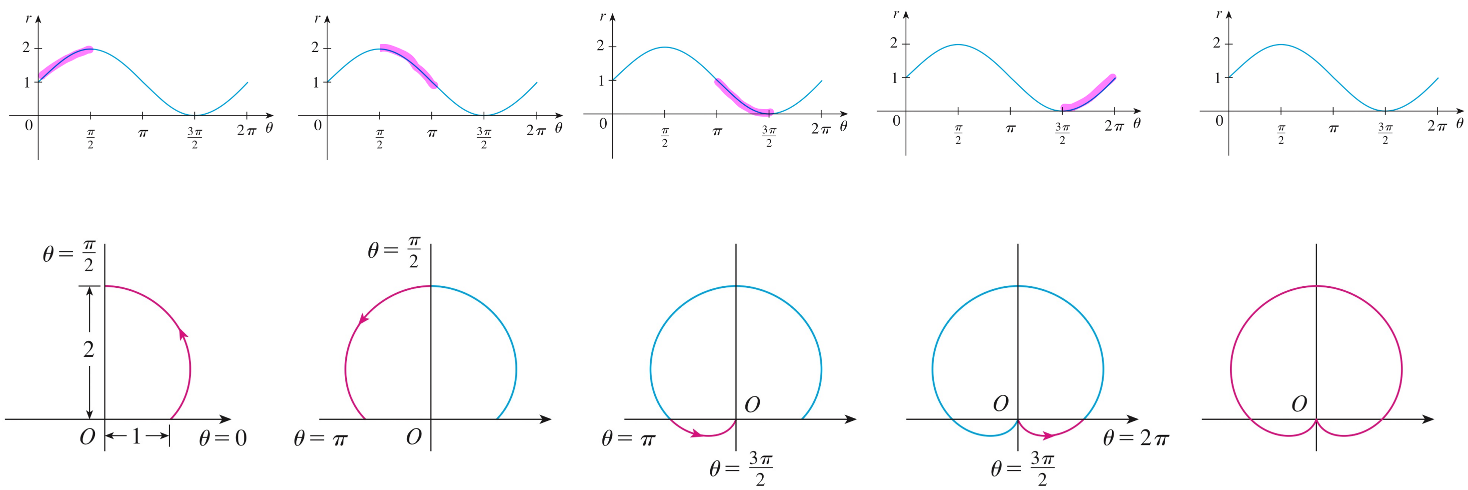


FIGURE 10
$r = 1 + \sin\theta$ in Cartesian coordinates,
$0 \leqslant \theta \leqslant 2\pi$





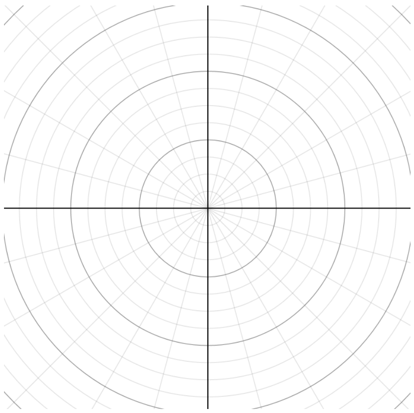
(a) (b) (c) (d) (e)

FIGURE 11 Stages in sketching the cardioid $r = 1 + \sin\theta$

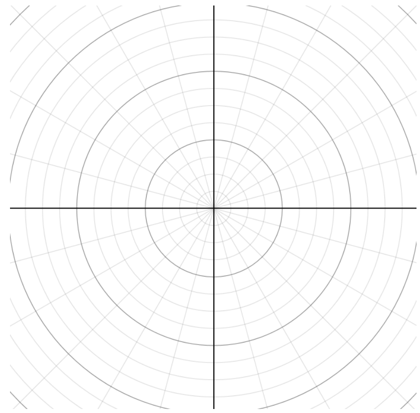
Lecture 10.3 **Polar Coordinates**

The graph of a polar equation $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

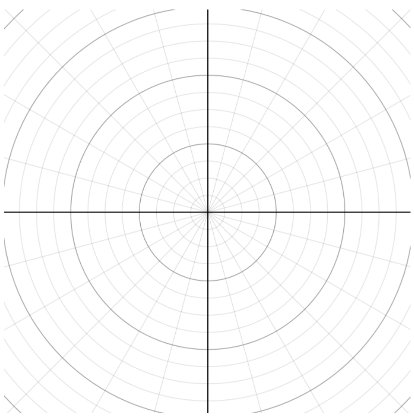
Exercise Sketch the following polar equations.



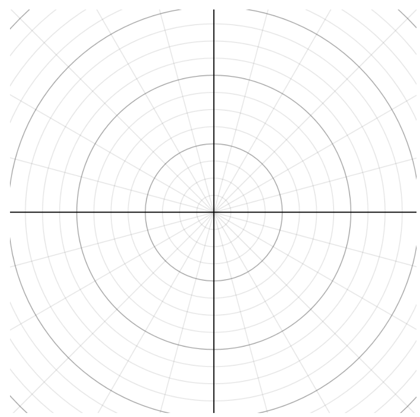
(a) $r = 4$



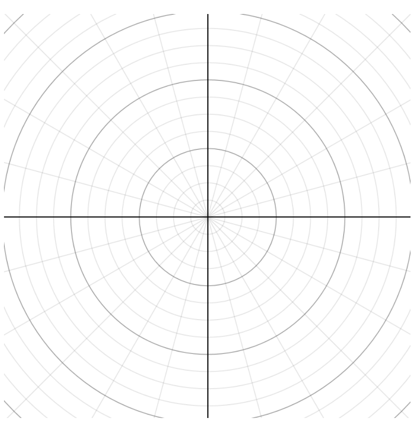
(b) $\theta = \dfrac{\pi}{6}$



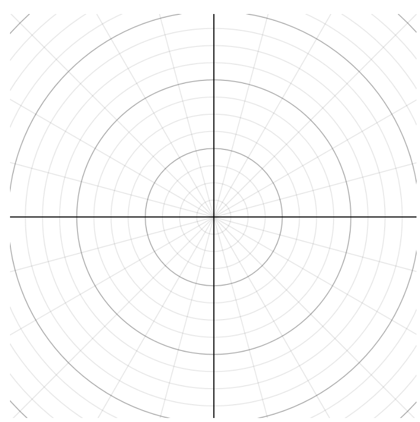
(c) $r = 4\cos\theta$



(d) $r = 4\sin\theta$



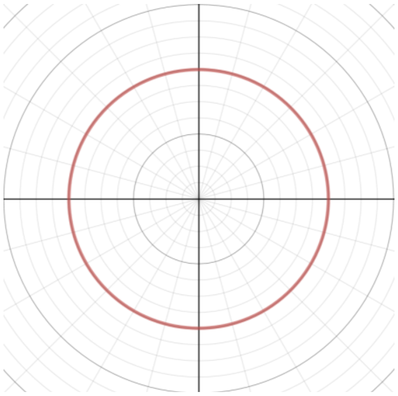
(e) $r = 2 + 2\sin\theta$



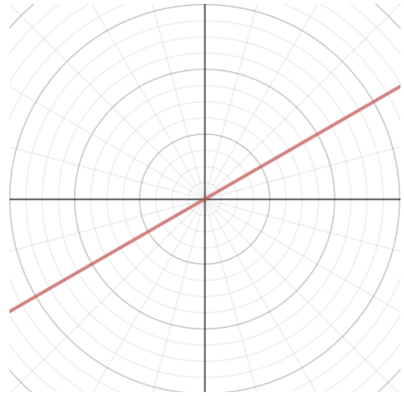
(f) $r = 4\cos 2\theta$

The graph of a polar equation $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

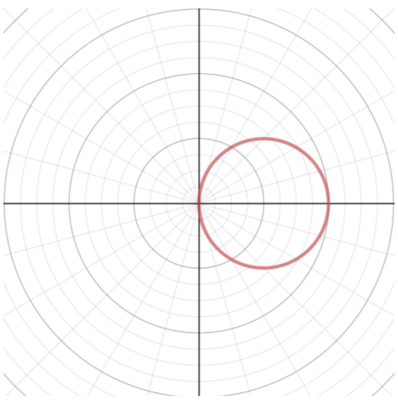
<u>Exercise (Answer Key)</u> Sketch the following polar equations.



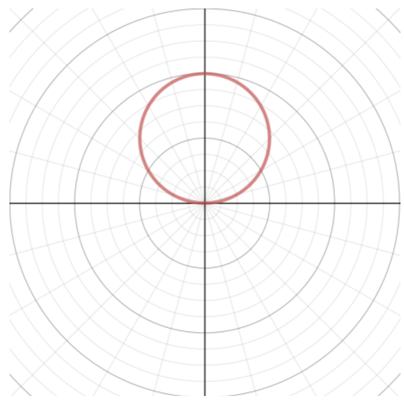
(a) $r = 4$



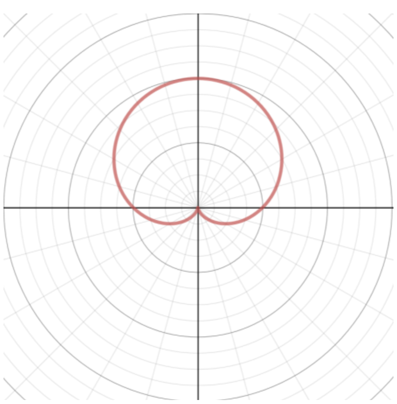
(b) $\theta = \dfrac{\pi}{6}$



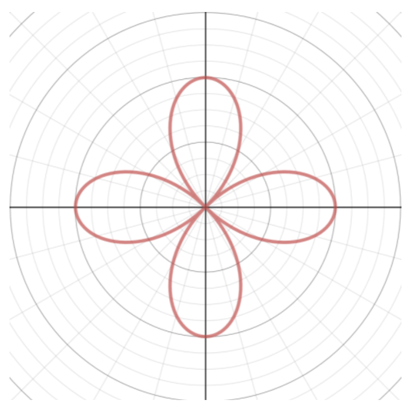
(c) $r = 4\cos\theta$



(d) $r = 4\sin\theta$



(e) $r = 2 + 2\sin\theta$
Cardioid



(f) $r = 4\cos 2\theta$
Four-leaved Rose

Recall tangents from Sec 10.2

Example

polar equation

For the cardioid $\underbrace{r = 1 + \sin\theta}$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.

Solution

$$x = r\cos\theta = \underbrace{(1 + \sin\theta)}_{r}\cos\theta = \cos\theta + \tfrac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = \underbrace{(1 + \sin\theta)}_{r}\sin\theta = \sin\theta + \sin^2\theta$$

Then we have Sec 10.2

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\ \cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)}{-\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right)} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2} - \frac{1}{2}} = \boxed{-1}$$