Lecture 10.1 **Curves Defined by Parametric Equations**

Idea

Imagine a particle moving along a curve C. The x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve, which we called a **parametric curve**.

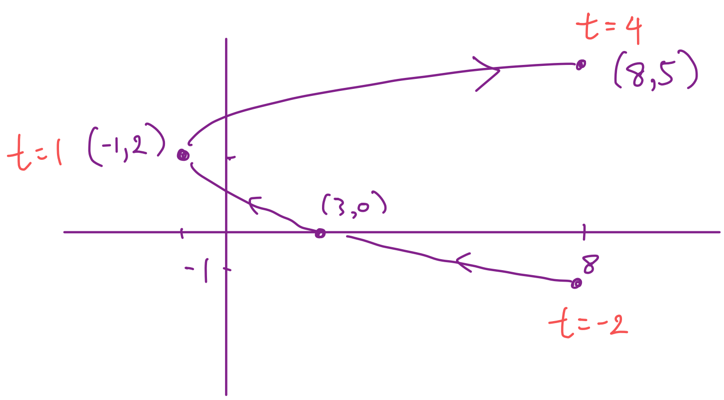
Webwork Problem 13, 17

Ex 1

Sketch and identify the curve defined by the parametric equations $\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}$ for $-2 \le t \le 4$

Plot

t	(x, y)
-2	$(8, -1)$
-1	$(3, 0)$
0	$(0, 1)$
1	$(-1, 2)$
2	
3	
4	



It appears that the curve traced out by the particle may be a parabola.

We can confirm this by eliminating the parameter: $y = t + 1 \Rightarrow t = y - 1$

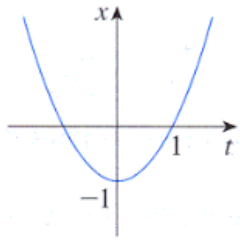
(Plug into $x = t^2 - 2t$) $x = (y-1)^2 - 2(y-1) \Rightarrow \boxed{x = y^2 - 4y + 3}$

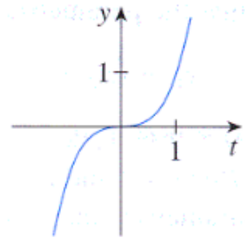
Visualize with Desmos: https://www.desmos.com/calculator/zt7cduetlf ← Click on link

Webwork Problem 12, 17

Ex 2

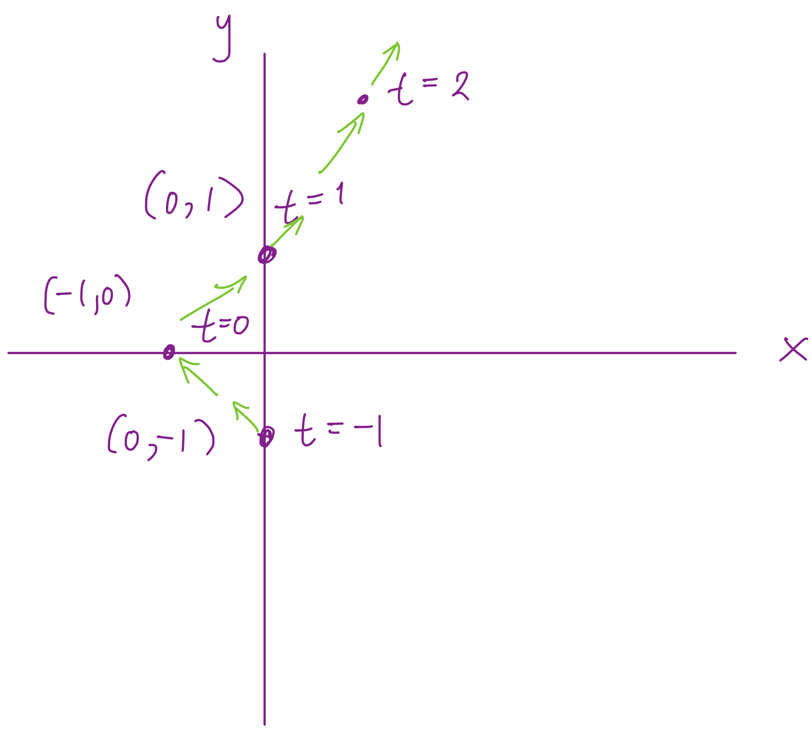
Given the graphs of parametric equations $x = f(t)$ and $y = g(t)$, sketch the parametric curve.





Plot

t	(x, y)
-1	$(0, -1)$
0	$(-1, 0)$
1	$(0, 1)$
2	(pos #, bigger than 1)



Visualize with Desmos: https://www.desmos.com/calculator/hhytfqheu9 ← Click on link

Curves Defined by Parametric Equations

Webwork Problem 5, 6, 7, 8:

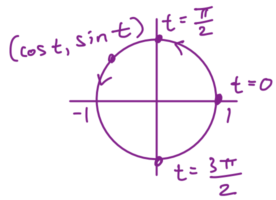
Ex 3 The circle $(x-6)^2+(y-8)^2=25$ is centered at $(6,8)$ and has radius 5.

The circle can be drawn with parametric equations using cosine and sine.

(a) If $x = 6 + 5\cos(t)$ and if we want to trace the circle once clockwise for $0 \le t \le 2\pi$ starting from the point $(11, 8)$, then y = ?

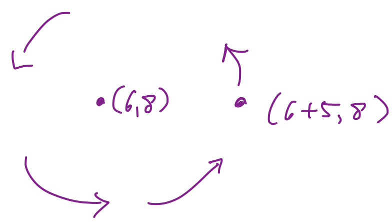
Answer

Method 1 (Unit circle):



$$\left.\begin{aligned} x &= 6+5\cos t \\ y &= 8+5\sin t \end{aligned}\right\} \text{ for } 0 \le t \le 2\pi$$

traces the circle



from $(6+5, 8)$ counter clockwise.

To reverse the direction, we can replace t with $-t$

$x = 6 + 5\cos(-t) = 6 + 5\cos t$

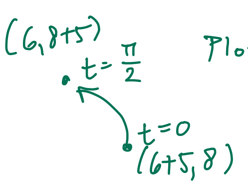
$\boxed{y = 8 + 5\sin(-t) = 8 - 5\sin t}$ ← answer

Answer

Method 2 (algebra):

- Plug in $x = 6+5\cos t$ into $(x-6)^2+(y-8)^2 = 25$:
  $x - 6 = 5\cos t$
  $25(\cos t)^2 + (y-8)^2 = 25$
- Use $(\cos t)^2 = 1-(\sin t)^2$:
  $25(1-(\sin t)^2) + (y-8)^2 = 25$
  $25 - 25(\sin t)^2 + (y-8)^2 = 25$
  $(y-8)^2 = 25(\sin t)^2$

So $y - 8 = 5\sin t$, $y = 8 + 5\sin t$ OR $\boxed{\begin{aligned} y-8 &= -5\sin t \\ y &= 8-5\sin t \end{aligned}}$ ← choose this because the question asks for clockwise



Plot:

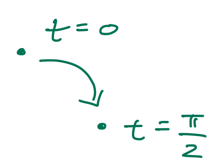
$t=0$	$x = 6+5,\ y = 8$
$t=\frac{\pi}{2}$	$x = 6,\ y = 8+5$

counterclockwise

Plot:

$t=0$	$x = 6+5,\ y = 8$
$t=\frac{\pi}{2}$	$x = 6,\ y = 8-5$

clockwise



Visualize with Desmos: https://www.desmos.com/calculator/108trp9mwd ← Click on link

Webwork Problem 5, 6, 7, 8 (con't):

Ex 3 The circle $(x-6)^2+(y-8)^2=25$ is centered at $(6,8)$ and has radius 5.

(b)

Write down a pair of parametric equations to describe a path *halfway* around the same circle counterclockwise starting from the point (6,13).

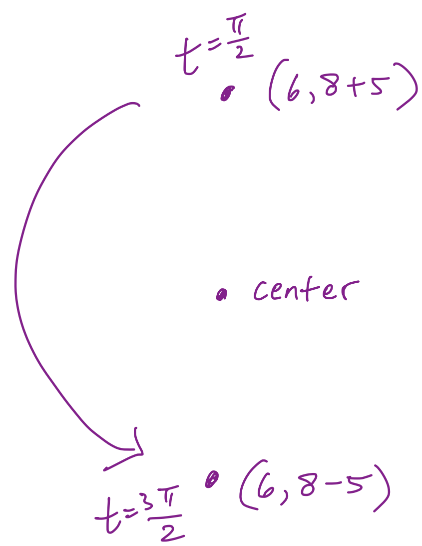
Answer we can use the original pair of equations

$$\left.\begin{aligned} x &= 6+5\cos t \\ y &= 8+5\sin t \end{aligned}\right\} \quad \text{for } \frac{\pi}{2} \le t \le \frac{3\pi}{2}$$

To make sure the path starts at $(6,13) = (6, \underbrace{8+5)}_{\text{the top of the circle}}$

start at $t = \frac{\pi}{2}$.

The path goes halfway around, so end at $t = \frac{3\pi}{2}$.



Visualize with Desmos: https://www.desmos.com/calculator/108trp9mwd

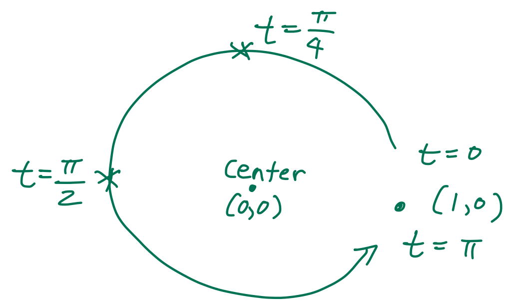
Webwork Problem 5, 6, 7, 8 (con't):

Ex 4

Problem:

Describe the parametic curve

$$\left.\begin{aligned} x &= \cos(2t) \\ y &= \sin(2t) \end{aligned}\right\} \quad \text{for} \quad 0 \le t \le \pi$$



Answer The path from $(x,y) = (1,0)$ around the unit circle one full circle going counterclockwise

Visualize with Desmos: https://www.desmos.com/calculator/108trp9mwd