① Memorize the statement of the Limit Comparison Test (for when $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive number)

② For $a_n = \frac{7}{5^n - 2}$, find a sequence b_n so that

$\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive (finite) number

Then apply the Limit Comparison Test to $\sum_{n=1}^{\infty} a_n$.

Answer

- Since a_n looks like $\frac{1}{5^n}$ and $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ is a geometric series,

  try $b_n := \frac{1}{5^n}$.

- $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{7}{5^n - 1} \cdot 5^n \overset{\text{by l'Hospital's Rule } \frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{7 \ln(5)\ 5^n}{\ln(5)\ 5^n} = 7$$

- Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 7$ is a positive number,

  either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both diverge.

- We know $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ converges (since it's a geometric series with ratio $\frac{1}{5}$ (living in $(-1,1)$), so $\sum_{n=1}^{\infty} a_n$ also $\boxed{\text{converges}}$.

Sec 11.4 Review pg 2

(3) Use the Limit Comparison Test to test $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^8-6}}$ for convergence/divergence.

Answer

Let $a_n := \dfrac{n}{\sqrt{n^8-6}}$.

Try $b_n := \dfrac{n}{\sqrt{n^8}} = \dfrac{n}{n^{\frac{8}{2}}} = \dfrac{n}{n^4} = \dfrac{1}{n^3}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^8+6}} \cdot n^3 = \lim_{n\to\infty} \frac{n^4}{\sqrt{n^8+6}} = \lim_{n\to\infty} \frac{\left(\frac{n^4}{n^4}\right)}{\sqrt{\frac{n^8}{n^8}+\frac{6}{n^8}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{6}{n^8}}} = 1$$

Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, a positive number,

the Limit Comparison Test says that either

$\left(\displaystyle\sum_{n=2}^{\infty} a_n \text{ and } \sum_{n=2}^{\infty} b_n \text{ both converge}\right)$ OR $\left(\displaystyle\sum_{n=2}^{\infty} a_n \text{ and } \sum_{n=2}^{\infty} b_n \text{ both diverge}\right)$.

Since $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^3}$ is a convergent p-series (due to $p=3>1$),

$\displaystyle\sum_{n=2}^{\infty} a_n$ also $\boxed{\text{converges}}$.

(4) Using the **Limit Comparison Test**, determine if the series $\sum_{n=1}^{\infty} \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5}$ converges.

Answer

Step 0 (Brainstorm).

- Dominant term of the top function: n^4
- Dominant term of the bottom function: $2n^6$ or n^6 $\quad \frac{n^4}{n^6} = \frac{1}{n^2}$
- So, try comparing this series with a p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ where p=2

Step 1. Let $a_n = \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5}$, $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^6 - 2n^4 + 3n^2}{2n^6 - n + 5} = \lim_{n\to\infty} \frac{n^6}{2n^6} = \frac{1}{2} > 0$$

Step 2. Since $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive, the series $\sum_{n=1}^{\infty} \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5}$ converges by the Limit Comparison Test

(since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series)

---

(5) Is $\sum_{n=1} \frac{\sqrt{n+1}}{n}$ convergent or divergent?

Answer

Let $a_n = \frac{\sqrt{n+1}}{n}$. Try $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

(i) $a_n = \frac{\sqrt{n+1}}{n} \geq \frac{\sqrt{n}}{n} = b_n$ for all $n = 1, 2, 3, \ldots$

(ii) $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series $(p = \frac{1}{2} \leq 1)$

So $\sum_{n=1}^{\infty} a_n$ diverges by the Comparison Test.

Sec 11.4 Review pg 4

(6) Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

---

(7) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{n^2}{n^{5/2}} = \frac{1}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot n^{1/2} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{\sqrt{5 + n^5}}$$

$$= \lim_{n\to\infty} \frac{2 + \frac{3}{n}}{\sqrt{\frac{5}{n^5} + 1}} = \frac{2 + 0}{\sqrt{0 + 1}} = 2 > 0$$

Since $\Sigma\, b_n = 2\,\Sigma\, 1/n^{1/2}$ is divergent $\left(p\text{-series with } p = \frac{1}{2} < 1\right)$, the given series diverges by the Limit Comparison Test. ■

Sec 11.5 Review Pg 1

Memorize the statement of the Alternating Series Test pg 772

Determine if the following series converge or diverge, with justification.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (Copy solution from Example 1, pg 774)

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ (Copy solution from Example 3, pg 774)

(c) $\sum_{n=1}^{\infty} (-1)^{n} \frac{3n}{4n-1}$ (Copy solution from Example 2, pg 774)

(d)

Determine whether the series $\sum_{n=1}^{\infty} \frac{-n^2 \cos(n\pi)}{n^3+1}$ converges/diverges.

Answer: Since $\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

we have $\cos(n\pi) = (-1)^n$ for all integers n

so the series is $\sum_{n=1}^{\infty} \frac{-n^2(-1)^n}{n^3+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

This is the series from Textbook Example 3, pg 774.

Copy the textbook's solution.

(e)

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{-n}$ converges/diverges.

Answer: Since $\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

we have $\cos(n\pi) = (-1)^n$ for all integers n

so the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{-n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

This is the series from Textbook Example 1, pg 774.

Copy the textbook's solution.

(f) Is $\sum_{n=2}^{\infty} \frac{(\sqrt{n}-1)\cos(n\pi)}{\sqrt{n^2-3}}$ an alternating series?

Answer yes, the series is $\sum_{n=2}^{\infty} \frac{(\sqrt{n}-1)(-1)^n}{\sqrt{n^2-3}}$ and $\frac{\sqrt{n}-1}{\sqrt{n^2-3}} > 0$ for all $n \geq 2$.

1.) Memorize the statement of the Ratio Test pg 779

Determine whether each of the following is conditionally convergent, absolutely convergent, or divergent.

2.) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Answer

- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges by Alternating Series Test

  since $\frac{1}{n^2}$ is decreasing for all $n = 1, 2, 3, \ldots$

  and $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

- $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2)$,

  so we say $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

3.) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

Answer

- Since (i) $\left\{ \frac{1}{\sqrt{n}} \right\}_{n=1}^{\infty}$ is decreasing

  (ii) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$,

  the series converges by Alternating Series Test.

- But $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent p-series $(p = \frac{1}{2})$

  So we say $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent.

Sec 11.6 Review pg 3

Determine whether each of the following is conditionally convergent, absolutely convergent, or divergent.

4.) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Solution

- $\sum \frac{\sin n}{n^2}$ has both positive and negative terms, but $\sum \frac{\sin n}{n^2}$ is not an alternating series (so Alternating Series Test doesn't apply)
- (We can apply comparison Test to $\sum \left|\frac{\sin n}{n^2}\right|$ with $\sum \frac{1}{n^2}$)

- Let $a_n := \frac{|\sin n|}{n^2}$ and $b_n := \frac{1}{n^2}$
- Since (i) $0 \le a_n \le b_n$ for all $n = 1, 2, 3, \ldots$ and (ii) $\sum b_n$ is a convergent p-series $(p = 2)$,

  $\sum \left|\frac{\sin n}{n^2}\right|$ also converges by the Comparison Test.
- By def, $\sum \frac{\sin n}{n^2}$ absolutely converges.

Sample answer to follow

- (meaning $\sum \frac{\sin n}{n^2}$ converges and $\sum \left|\frac{\sin n}{n^2}\right|$ converges).

5.) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$

Answer

- $$\frac{a_{k+1}}{a_k} = \frac{\left(\frac{10^{k+1}}{(k+1)!}\right)}{\left(\frac{10^k}{k!}\right)} = \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k} = \frac{10}{k+1}$$
- $$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{10}{k+1} = 0$$
- $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$ is absolutely convergent

  by the Ratio Test, since $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 0 < 1$.

WEBWORK PROBLEM 1, 2, 7, 8, 11, 12

Sec 11.6 Review pg 4

Determine whether each of the following is conditionally convergent, absolutely convergent, or divergent.

6.) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent.

7.) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e \quad \text{as } n \to \infty$$

See Exam 1 sol for Sec 6.8 l'Hospital's Rule

(see ~~Equation 6.4.9 or 6.4*.9~~). Since $e > 1$, the given series is divergent by the Ratio Test. ■
Exam 1 solution for Sec 6.8 question

Sec 11.7 Review pg 1

For each of the following series, determine whether the ratio test will work for testing convergence / divergence.

① $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{(n^2+4)}$

- ○ The ratio test can be used
- ● The ratio test will be inconclusive

② $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^5}$

- ○ The ratio test can be used
- ● The ratio test will be inconclusive

③ $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(\pi n)+6\sqrt{n}}{n^2}$

- ○ The ratio test can be used
- ● The ratio test will be inconclusive

④ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+1)!\,2}$

- ○ The ratio test can be used
- ● The ratio test will be inconclusive

⑤ $\displaystyle\sum \frac{(-2)^n}{3^n}\frac{\ln n}{(n^2+4)}$

- ● The ratio test can be used
- ○ The ratio test will be inconclusive

⑥ $\displaystyle\sum \frac{\sqrt{n}+1}{n!}(-1)^n$

- ● The ratio test can be used
- ○ The ratio test will be inconclusive

⑦ $\displaystyle\sum_{n=2}^{\infty} \frac{6}{n\sqrt{n^7-2}}$

- ○ The ratio test can be used
- ● The ratio test will be inconclusive

Sec 11.7 Review pg 2

Key points from textbook pg 784-785

(8)

- Q: When will Ratio Test be inconclusive ?

A: If the terms of the series involve n^p and $\ln(n)$ only

e.g $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{(n^2+4)}$ or $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^5}$ or $\sum_{n=1}^{\infty} \frac{\ln(\pi n)+6\sqrt{n}}{n^2}$

(9)

- Q: When is Ratio Test likely to work ?

A: If the terms involve factorial, geometric sequence, n^n

e.g $\sum \frac{(-2)^n}{3^n} \frac{\ln n}{(n^2+4)}$ or $\sum \frac{\sqrt{n}+1}{n!} (-1)^n$

(10)

- Q: When can Limit Comparison Test work ?

A: Series with only positive (or 0) terms

e.g $\sum_{n=1} \frac{\ln(\pi n)+6\sqrt{n}}{n^2}$ or $\sum_{n=1}^{\infty} \frac{2^n}{3^n} \frac{\ln n}{(n^2+4)}$ or $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^5-1}$ or $\sum_{n=2}^{\infty} \frac{6}{n\sqrt{n^7-2}}$

WEBWORK PROBLEM 1

WEBWORK PROBLEM 4

Sec 11.8 Review pg 1

Q: What theorem should you use to find the radius of convergence of a power series? A: Ratio Test

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Q: What tests should you use to check whether $x = a+R$ and $x = a-R$ (where a is the center of the power series and R is the radius) are in the interval of convergence?

A: p-series; geometric series; Alternating Series Test; Limit Comparison Test. (There are others, but you don't need other ways on this exam)

Q: If $\lim_{n\to\infty} \left| \frac{C_{n+1}(x-10)^{n+1}}{C_n (x-10)^n} \right| = \left| \frac{x-10}{99} \right|$, what is the radius of convergence for $\sum_{n=1}^{\infty} C_n (x-10)^n = c_1(x-10) + c_2(x-10)^2 + c_3(x-10)^3 + \dots$? What about interval of convergence?

A: $\sum_{n=1}^{\infty} C_n (x-10)^n$ converges when $\left|\frac{x-10}{99}\right| < 1 \Leftrightarrow |x-10| < 99$, so the radius of convergence is 99.
The interval of convergence includes $(10-99, 10+99)$, but we don't have enough information to determine if -89 and 109 are in the interval.

WEBWORK PROBLEM 2, 6, 9, 10

Sec 11.8 Review Pg 2

Find the radius of convergence and interval of convergence for each series.

1. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{5^n} = 1 + \frac{x+2}{5} + \frac{(x+2)^2}{25} + \dots$

Answer:

$\sum_{n=0}^{\infty} \left(\frac{x+2}{5}\right)^n$ is a geometric series with ratio $\frac{x+2}{5}$

So the series converges iff $\left|\frac{x+2}{5}\right| < 1 \iff |x+2| < 5$

$-5 < x+2 < 5$

$-5-2 < x < 5-2$

$-7 < x < 3$

Interval of convergence is $(-7, 3)$
Radius of convergence is $R = 5$

---

2. $\sum_{n=0}^{\infty} 9^n (x-2)^{2n}$

Answer

- $\sum_{n=0}^{\infty} 9^n (x-2)^{2n} = \sum_{n=0}^{\infty} \left[9(x-2)^2\right]^n$
- This is a geometric series with ratio $9(x-2)^2$.
- So it converges if and only if $|9(x-2)^2| < 1$

$|(x-2)^2| < \frac{1}{9}$

$|x-2| < \frac{1}{3}$

$-\frac{1}{3} < x-2 < \frac{1}{3}$

$2 - \frac{1}{3} < x < 2 + \frac{1}{3}$

Interval of convergence is $(1\frac{2}{3}, 2\frac{1}{3})$.

Radius of convergence is $\frac{1}{3}$.

3. Find the radius of convergence and interval of convergence for $\sum_{n=1}^{\infty} \frac{5^n (x-4)^n}{\sqrt{n}}$

Answer

- Let $a_n = \frac{5^n (x-4)^n}{\sqrt{n}}$

- $\frac{a_{n+1}}{a_n} = \frac{5^{n+1} (x-4)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{5^n (x-4)^n}$

$= \frac{5 \sqrt{n} (x-4)}{\sqrt{n+1}}$

- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{5\sqrt{n}}{\sqrt{n+1}} (x-4)\right|$

$= |5(x-4)| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}$

$= |5(x-4)| \cdot 1$

- By Ratio Test, $\sum a_n$ converges when $|5(x-4)| < 1$

$|x-4| < \frac{1}{5}$

$-\frac{1}{5} < x-4 < \frac{1}{5}$

- So the $\boxed{\text{radius of convergence is } \frac{1}{5}}$

- What is the interval of convergence I?

We know I must include $\left(4-\frac{1}{5}, 4+\frac{1}{5}\right)$

Check $x = 4-\frac{1}{5}$: $\sum_{n=1}^{\infty} \frac{5^n \left(-\frac{1}{5}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by Alternating Series Test
(so I includes $x = 4-\frac{1}{5}$)

Check $x = 4+\frac{1}{5}$: $\sum_{n=1}^{\infty} \frac{5^n \left(\frac{1}{5}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $\left(p=\frac{1}{2}\right)$
(So I does <u>not</u> include $x = 4+\frac{1}{5}$)

$\boxed{I = \left[4-\frac{1}{5}, 4+\frac{1}{5}\right)}$ is the interval of convergence

Sec 11.8 Review Pg 4

Find the radius of convergence and interval of convergence for each series.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Answer

Use Ratio Test:

- Let $a_n = \dfrac{x^n}{n!}$
- $\dfrac{a_{n+1}}{a_n} = \dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n} = \dfrac{1}{n+1} \cdot x$
- $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$ for any number x
- So by Ratio Test, $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

> Interval of convergence is $(-\infty, \infty)$
> Radius of convergence is ∞.

---

5. $\displaystyle\sum_{n=0}^{\infty} n!(x+8)^n$

Answer

Use Ratio Test:

- Let $a_n = n!\,(x+8)^n$
- $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!\,(x+8)^{n+1}}{n!\,(x+8)^n}$

  $= (n+1)(x+8)$
- $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|(n+1)(x+8)\right| = |x+8| \lim_{n\to\infty} (n+1) = \begin{cases} \infty & \text{if } x \neq -8 \\ 0 & \text{if } x = -8 \end{cases}$
- The series converges if and only if $x = -8$

> Interval of convergence is $\{-8\} = [-8, -8]$
> Radius of convergence is 0.

Sec 11.8 Review pg 5

6. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -3$ and diverges when $x = 4$.
Determine whether the following series converge or diverge.

1. $\sum_{n=1}^{\infty} c_n$

2. $\sum_{n=1}^{\infty} c_n 9^n$

3. $\sum_{n=1}^{\infty} c_n (-2)^n$

4. $\sum_{n=1}^{\infty} (-1)^n c_n 12^n$

Solution

Since the center of this power series is 0, this means the radius of convergence, R, is between 3 and 4 (possibly 3 or 4).

1. $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} c_n \; (1)^n$ Distance between center and x=1 is $1 < 3$
$\Rightarrow$ the series converges

2. $\sum_{n=1}^{\infty} c_n 9^n$ Distance between center (0) and x=9 is $9 > 4$
$\Rightarrow$ the series diverges

3. $\sum_{n=1}^{\infty} c_n (-2)^n$ Distance between center (0) and x=-2 is $2 < 3$
$\Rightarrow$ the series converges

4. $\sum_{n=1}^{\infty} (-1)^n c_n 12^n = \sum_{n=1}^{\infty} c_n \, (-12)^n$ Distance between center (0) and x=-12 is $12 > 4$
$\Rightarrow$ the series diverges

Sec 11.9 Review   Pg 1

(1) Find a power series representation for $f(x) = \dfrac{5}{1+4x^2}$ and find its interval of convergence.

Answer

$$\frac{5}{1+4x^2} = 5\,\frac{1}{1-(-4x^2)}$$

$$= \boxed{5\sum_{n=0}^{\infty}(-4x^2)^n}$$

if $|-4x^2| < 1 \iff |x^2| < \frac{1}{4}$

$\iff |x| < \underbrace{\tfrac{1}{2}}_{}$

radius of convergence

$$= \boxed{5\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}}$$

$\boxed{\text{Interval of convergence: } \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)}$

---

(2) Find a power series representation for $f(x) = \dfrac{2x^4}{2-3x}$ and find its interval of convergence.

Answer

$$\frac{\frac{1}{2}\,2x^4}{\frac{1}{2}(2-3x)} = \frac{x^4}{1-\frac{3}{2}x}$$

$$= x^4\,\frac{1}{1-\left(\frac{3}{2}x\right)}$$

Radius of convergence

$$= \boxed{x^4\sum_{n=0}^{\infty}\left(\frac{3}{2}x\right)^n}$$

if $\left|\frac{3}{2}x\right| < 1 \iff |x| < \frac{2}{3}$

$$= \boxed{x^4\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n x^n}$$

$$= \boxed{\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n x^{4+n}}$$

$\boxed{\text{Interval of convergence: } \left(-\tfrac{2}{3}, \tfrac{2}{3}\right)}$

Sec 11.9 Review Pg 2

3 Find a power series representation (centered at 0) for $f(x) = \dfrac{1}{(5+x)^2}$.

Answer

Step 0 $\dfrac{d}{dx}\left[\dfrac{1}{(5+x)}\right] = -\dfrac{1}{(5+x)^2}$

$$\frac{d}{dx}\left[-\frac{1}{5+x}\right] = \frac{1}{(5+x)^2}$$

Step 1 $-\dfrac{1}{5+x} = -\dfrac{\frac{1}{5}}{\frac{1}{5}(5+x)}$

$$= -\frac{1}{5}\,\frac{1}{1-\left(-\frac{x}{5}\right)}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n \quad \text{for } \left|-\frac{x}{5}\right| < 1 \iff |x| < 5$$

Radius of convergence: 5

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n x^n$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^{n+1} x^n$$

Step 2 $\dfrac{1}{(5+x)^2} = \dfrac{d}{dx}\left[-\dfrac{1}{5+x}\right]$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^{n+1} x^n\right]$$

$$= \frac{d}{dx}\left[\left(-\frac{1}{5}\right)x^0 + \sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)^{n+1} x^n\right]$$

$$= 0 + \boxed{\sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)^{n+1} n\,x^{n-1}}$$

by Thm (term-by-term differentiation)

(you can stop here)

$$= \boxed{\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^{n+2}(n+1)\,x^n}$$

Sec 11.9 Review Pg 3

(4) Find the antiderivative of the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots \quad \text{for } |x| < 1$$

Answer

$$\int \frac{1}{1+x}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^n \right) dx$$

$$\overset{(*)}{=} \int (1 - x + x^2 - x^3 + \ldots)\,dx$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right) + C$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

---

(5) Find $\int \ln(1+t^4)\,dt$ as a power series, and find its radius of convergence.

Answer

Step 1:

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^n \right) dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

from (*) above

To find C, plug in the center $x=0$ of the power series:

$$\ln(1+0) = 0 + C$$

so $C = 0$

Here, we have to solve for C.

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for } |x| < 1$$

Step 2:

So $$\ln(1+t^4) = \sum_{n=0}^{\infty} (-1)^n \frac{(t^4)^{n+1}}{n+1} \quad \text{for } |t^4| < 1 \iff |t| < 1$$

Radius of Convergence is 1

$$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+4}}{n+1}$$

Step 3:

$$\int \ln(1+t^4)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+5}}{(n+1)(4n+5)} + \text{Constant}$$

Same radius of convergence, 1

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(6) Use the fact $\arctan(x) \overset{(*)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ (with radius of convergence 1) to find a power series representation of $\int \frac{\arctan(2x)}{x}\,dx$. Find its radius of convergence.

← impossible to solve using Chapter 7 methods

Answer

$$\arctan(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} \quad \text{by } (*) \text{ given above}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{2n+1}$$

for $|2x| < 1 \iff |x| < \frac{1}{2}$

(Radius of convergence is $\frac{1}{2}$)

$$\frac{\arctan(2x)}{x} = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{2n+1} x^{2n}$$

$$\int \frac{\arctan(2x)}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{2n+1} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{2n+1} \frac{x^{2n+1}}{2n+1} + C$$

$$= \boxed{\left(2x - \frac{2^3 x^3}{9} + \frac{2^5 x^5}{25} - \frac{2^7 x^7}{49} + \dots\right) + C}$$

Radius of convergence is the same as for the series for $\arctan(2x)$: $\frac{1}{2}$

Sec 11.10 Review pg 1

The boxed equations are from Table 1, pg 808 (will be printed for you).

Fill in the blanks.

- $\boxed{\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } -1 \le x < 1.}$

1 ➢ $\sum_{n=1}^{\infty} \frac{1}{n\,2^n} = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n} = -\ln\left(1-\frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \ln(2)$

2 • $\ln(1+x) = \ln(1-(-x)) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ for $-1 < x \le 1$.

- $\boxed{\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{for } -1 \le x \le 1.}$

3 ➢ $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}$.

- $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } -\infty < x < \infty.}$

5 Find the sum of $1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \cdots$

Answer The series is $\sum_{n=0}^{\infty} \frac{4^n}{n!} = e^4$

4 ➢ $\sum_{n=0}^{\infty} \frac{1}{n!} = e^1$.

- $\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for } -\infty < x < \infty.}$

6 ➢ $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin(\pi) = 0$.

- $\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x}$.

7 $\sum_{n=0}^{\infty} \frac{(-1)^n}{6^{2n}} \frac{\pi^{2n}}{(2n)!} = \cos\left(\frac{\pi}{6}\right)$

Theorem (given for you)
IF f has a power series representation at $x = a$, that is, if

$$f(x) = \underbrace{\sum_{n=0}^{\infty} c_n (x-a)^n}_{\text{called "the Taylor series of } f(x) \text{ centered at } a\text{"}} \qquad \text{for} \qquad |x-a| < R,$$

THEN its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

8 If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and $f(0) = 14, \quad f'(0) = -15, \quad f''(0) = -1, \quad f'''(0) = -1$,

find the first four terms of $\sum_{n=0}^{\infty} c_n x^n$.

Answer $c_0 = \frac{f(0)}{0!} = 14 \qquad c_1 = \frac{f'(0)}{1!} = -15 \qquad c_2 = \frac{f''(0)}{2!} = -\frac{1}{2} \qquad c_3 = \frac{f'''(0)}{3!} = -\frac{1}{6}$

$$\boxed{14 - 15x - \frac{1}{2}x^2 - \frac{1}{6}x^3}$$

---

9 (From Textbook Example 1) If $f^{(n)}(a) = 3^n$ for all $n \geq 0$, what is the Taylor series of $f(x)$ centered at $x = a$?

Answer $f(a) = 3^0, \; f'(a) = 3^1, \; f''(a) = 3^2, \ldots$

are given, so $c_n = \frac{f^{(n)}(a)}{n!} = \frac{3^n}{n!}$

So the answer is $\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{3^n}{n!}(x-a)^n$

$$= 1 + 3(x-a) + \frac{3^2}{2}(x-a)^2 + \frac{3^3}{3!}(x-a)^3 + \frac{3^4}{4!}(x-a)^4 + \ldots$$

Sec 11.10 Review pg 3

10 • One of the Webwork problems is to find that

$$\boxed{\ln(x) = \ln(10) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{10^n n}(x-10)^n}$$

when x is in the interval of convergence of the power series.

• Assume you have done the computation showing this equality.

Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{10^n n}(x-10)^n$.

Answer

Step 1 (Find radius of convergence using Ratio Test)

Let $a_n = \frac{(-1)^{n-1}}{10^n n}(x-10)^n$

$$\frac{a_{n+1}}{a_n} = \frac{(x-10)^{n+1}}{10^{n+1}(n+1)} \cdot \frac{10^n n}{(x-10)^n} = \frac{(x-10)}{10}\frac{n}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x-10}{10}\frac{n}{n+1}\right| = \left|\frac{x-10}{10}\right| \lim_{n\to\infty}\frac{n}{n+1} = \left|\frac{x-10}{10}\right|$$

By Ratio Test, $\sum_{n=1}^{\infty}|a_n|$ converges when $\left|\frac{x-10}{10}\right| < 1$

$\iff |x-10| < 10$ This is the radius of convergence

$-10 < x-10 < 10$

$10-10 < x < 10+10$

Step 2 (Check endpoints)

Check $x = 10-10 = 0$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n}(-10)^n = \sum_{n=0}^{\infty} (-1)^n(-1)^n \frac{10^n}{10^n}\frac{1}{n}$

$= \sum_{n=0}^{\infty} \frac{1}{n}$ is the Harmonic series (divergent)

Check $x = 10+10 = 20$: $\sum_{n=0}^{\infty} \frac{(-1)^n (10)^n}{10^n n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$ is convergent

by Alternating Series Test

($\left\{\frac{1}{n}\right\}$ is a decreasing sequence $\frac{1}{n} > \frac{1}{n+1}$ for all $n \geq 0$)

Interval of Convergence: $\boxed{(0, 20]}$

Sec 11.10 Review pg 4

Table 1

Important Maclaurin Series and Their Radii of Convergence

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$
$\text{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$R = 1$

11

(i) Evaluate $\int \frac{1}{2x}\,dx$ (write as a "usual" elementary function)

12

(ii) Use part (i) and Table 1 above to write $\int \frac{e^{9x}-1}{2x}\,dx$ as a power series

(i) $\boxed{\int \frac{1}{2x}\,dx = \frac{1}{2}\ln x + C_1}$

(ii) $e^{9x} = \sum_{n=0}^{\infty} \frac{(9x)^n}{n!} = \sum_{n=0}^{\infty} 9^n \frac{1}{n!} x^n$ from Table 1, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\frac{e^{9x}}{2x} = \frac{1}{2}\sum_{n=0}^{\infty} 9^n \frac{1}{n!} x^{n-1} = \frac{1}{2}\left[9^0 \frac{1}{0!}\frac{1}{x} + \sum_{n=1}^{\infty} 9^n \frac{1}{n!} x^{n-1}\right]$$

$$\int \frac{e^{9x}}{2x}\,dx = \frac{1}{2}\left[\ln x + \sum_{n=1}^{\infty} 9^n \frac{1}{n!}\frac{x^n}{n}\right] + C_2$$

So $$\int \frac{e^{9x}-1}{2x}\,dx = \int \frac{e^{9x}}{2x}\,dx - \int \frac{1}{2x}\,dx$$

$$= \frac{1}{2}\left[\ln x + \sum_{n=1}^{\infty} 9^n \frac{1}{n!}\frac{x^n}{n}\right] + C_2 - \left[\frac{1}{2}\ln x + C_1\right]$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{9^n}{2}\frac{1}{n!}\frac{1}{n} x^n + C}$$

Sec 11.10 Review pg 5

13. Use the table to write a power series representation of $\int \arctan(x^2)\,dx$ centered at $x=0$.

$$\arctan(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan(x^2)\,dx = \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}}$$

14. Use the table to write a power series representation of $\int \arctan(x^2)\,dx$ centered at $x=0$.

$$x^3 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{2n+1}$$

$$\int x^3 \arctan x = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{2n+1} = \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+5}}{(2n+1)(2n+5)}}$$

15. Use Table 1 to write $\int x^2 \sin(x^2)\,dx$ as a power series.

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$x^2 \sin(x^2) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!}$$

$$\int x^2 \sin(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!}\,dx = \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(2n+1)!\,(4n+5)}}$$

16. Use Table 1 to write $\int \sin(x^4)\,dx$ as a power series centered at $x=0$.

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, so $\sin(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}$ for all x and

$$\int \sin(x^4)\,dx = \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+5}}{(2n+1)!\,(8n+5)}}.$$

Statements to memorize

☐ Alternating Series Test — Sec 11.5

Statement:

☐ Limit Comparison Test — Sec 11.4

Statement:

☐ Ratio Test — Sec 11.6

Statement:

Sec 11.2

○ a geometric series converges

when ...

○ a geometric series diverges

when ...

Sec 11.4

○ a p-series series converges

when...

○ a p-series series diverges

when...