# Sec 7.1 Review Part I

Look up solutions on...

---

pg 513

- Find $\int x \sin(x)\, dx$
- Evaluate $\int \ln(x)\, dx$

---

pg 514

- Compute $\int x e^x\, dx$. Use it to compute $\int x^2 e^x\, dx$
- Calculate $\int e^x \sin(x)\, dx$.

---

pg 515

- Compute $\int_0^1 \frac{x}{1+x^2}\, dx$
- Calculate $\int_0^1 \arctan(x)\, dx$

---

Lecture notes 7.1

- Calculate $\int_0^1 \arcsin(x)\, dx$

---

7.1 Recommended Textbook Practice SOLUTIONS.pdf

- Evaluate
  - (#2) $\int \sqrt{x}\, \ln(x)\, dx$
  - (#15) $\int (\ln x)^2\, dx$
  - (#26) $\int_1^2 x^2 \ln(x)\, dx$
  - (#27) $\int_1^5 \frac{\ln(x)}{x^2}\, dx$
  - (#31) $\int_1^5 \frac{x}{e^x}\, dx$
  - (#58) $\int_1^0 \left[x e^{-x} - x^2 e^{-x}\right] dx$
  - (#61) $\int_0^1 2\pi x \cos\left(\frac{\pi}{2} x\right) dx$

# Webwork 7.1

If $g(1) = 3$, $g(5) = 10$, and $\int_1^5 g(x)\,dx = -10$, evaluate the integral $\int_1^5 xg'(x)\,dx$.

Let $u = x$, $dv = g'(x)\,dx \Rightarrow du = dx$, $v = g(x)$. We then have

$$\int_1^5 xg'(x)\,dx = \underbrace{\Big[xg(x)\Big]_1^5 - \int_1^5 g(x)\,dx}_{uv\big|_1^5 - \int_1^5 v\,du} = \Big[5 \cdot g(5) - 1 \cdot g(1)\Big] + \underbrace{10}_{\text{given}} = 5 \cdot (10) - 3 + 10 = 57$$

$u = x \quad dv = g'(x)\,dx$

$du = dx \quad v = g(x)$

---

# Webwork 7.1

Suppose that $f(1) = -5$, $f(4) = -2$, $f'(1) = -7$, $f'(4) = 1$, and $f''$ is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

$$\int x\, f''(x)\,dx = uv - \int v\,du = x\,f'(x) - \int f'(x)\,dx$$

| $u = x$ | $dv = f''(x)\,dx$ |
|---|---|
| $du = dx$ | $v = f'(x)$ |

$$\int_1^4 x\, f''(x)\,dx = x\,f'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx$$

$$= 4f'(4) - 1f'(1) - \Big[f(4) - f(1)\Big]$$

$$= 4(1) - (-7) - \Big[(-2) - (-5)\Big]$$

$$= 4 + 7 - [3]$$

$$= \boxed{8}$$

# Sec 7.2 Review

These identities will be given if needed:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\qquad$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin x \cos x = \frac{1}{2}\sin 2x$

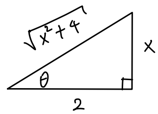
Look up solutions on ...

| | |
|---|---|
| Strategy for $\int (\sin x)^{\text{even number}}\,dx$ ?<br>Strategy for $\int (\cos x)^{\text{even number}}\,dx$ ? | pg 521 |
| Compute $\int (\sin x)^4\,dx$, $\int_0^{\pi} (\sin x)^2\,dx$, $\int (\cos x)^2\,dx$ | pg 520 |
| Strategy for $\int (\sin x)^{\text{odd number}} (\cos x)^m\,dx$ ?<br>Strategy for $\int (\cos x)^{\text{odd number}} (\sin x)^n\,dx$ ? | pg 521 |
| Compute $\int (\cos x)^3\,dx$, $\int (\sin x)^5 (\cos x)^2\,dx$ | pg 519 |
| Compute $\int (\cos x)^7\,dx$, $\int (\sin x)^7\,dx$ | Lecture notes 7.2 |
| Compute $\int \tan x\,dx$ | Old Section 6.4 pg 432 |
| Compute $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$ | Apply u-sub $u = \sin x$, $du = \cos x\,dx$ |

# Sec 7.3 Review Part I

Look up solutions on ...

---

If 


then $x = \boxed{2\tan\theta}$ $\qquad \dfrac{2}{\sqrt{x^2+4}} = \boxed{\cos\theta}$

$dx = \boxed{2(\sec\theta)^2\,d\theta}$

$\sqrt{x^2+4} = \boxed{2\sec\theta}$

Lec Notes 7.3 B

---

Set up the trig substitution for $\displaystyle\int_0^2 x^3\sqrt{x^2+4}\,dx$.

Stop there. DO NOT solve the integral.

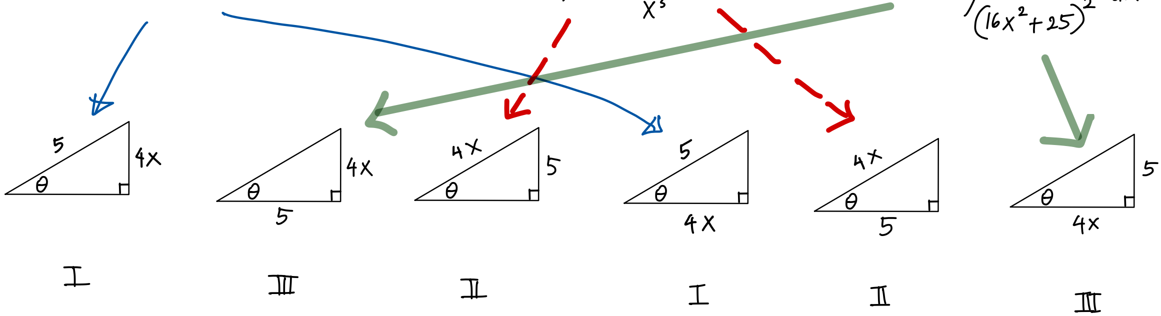
Lec Notes 7.3 B

**Answer** $\displaystyle\int_{\theta=0}^{\theta=\frac{\pi}{4}} 8(\tan\theta)^3\ \frac{2}{\cos\theta}\ 2(\sec\theta)^2\,d\theta = 32\int_0^{\frac{\pi}{4}} (\tan\theta)^3(\sec\theta)^3\,d\theta$

---

For each integral, choose all triangle/s that will work for trig substitution

I $\displaystyle\int x^2\sqrt{25-16x^2}\,dx$ $\qquad$ II $\displaystyle\int \frac{\sqrt{16x^2-25}}{x^3}$ $\qquad$ III $\displaystyle\int \frac{1}{(16x^2+25)^2}\,dx$




I : $\sqrt{25}$ must be the hypotenuse.

$\sqrt{25-16x^2}$ can be either the opp or adj.

II : $\sqrt{16x^2}$ must be the hypotenuse.

$\sqrt{16x^2-25}$ can be either the opp or adj

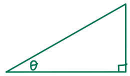
III : $\sqrt{16x^2+25}$ must be the hypotenuse

4x and 5 are opp / adj (doesn't matter)

# Sec 7.3 Review Part II

Look up solutions on ...

Compute $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$ using trig substitution.

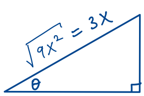
Label all three sides of 

so that they match your trig sub.

Lec Notes 7.3 A

---

Compute $\displaystyle\int \frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}}\,dx$ by performing trig substitution

using 

. (Don't use other methods)

Lec Notes 7.3 C

# Sec 7.4 Review Part I

What is the partial fraction decomposition for $\frac{9x+3}{x^2-9}$ ?

(a) $\frac{A}{x-3} + \frac{B}{x+3}$

$\frac{9x+3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$ by Partial Fraction Theorem CASE I

(b) $\frac{Ax+B}{x-3} + \frac{Cx+D}{x+3}$

(c) $\frac{A}{x^2} + \frac{B}{9}$

(d) $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$

---

What is the partial fraction decomposition for $\frac{5x^3-3x^2-8x-3}{x^4-3x^3}$

(a) $\frac{A}{x^3} + \frac{B}{x-3}$

(b) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$

$$\frac{5x^3-3x^2-8x-3}{x^3(x-3)} = \underbrace{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}}_{\text{because factor } x \text{ has 3 repetitions}} + \frac{D}{x-3}$$

(c) $\frac{A}{x} + \frac{B}{x-3}$

(d) $\frac{A}{x^4} + \frac{B}{3x^3}$

---

Helpful fact given: $\frac{6t}{(3t-5)(3t+5)} = \frac{1}{3t-5} + \frac{1}{3t+5}$ by partial fraction decomposition.

Question Evaluate $\int \frac{6x}{9x^2-25}\,dx$

$\boxed{\frac{6x}{9x^2-25} = \frac{1}{3x-5} + \frac{1}{3x+5} \text{ (given above – you don't need to compute)}}$

$$\int \frac{6x}{9x^2-25}\,dx = \int \left[\frac{1}{3x-5} + \frac{1}{3x+5}\right] dx$$

$$= \frac{\ln|3x-5|}{3} + \frac{\ln|3x+5|}{3} + C$$

# Sec 7.4 Review part II

True or false? It is possible to integrate every rational function (a ratio of polynomials) in terms of the functions we know

**TRUE**

- If a rational function $\frac{P(x)}{Q(x)}$ is proper, meaning $\deg(P) < \deg(Q)$ then it falls into CASES I – IV (see top of pg 534)
- If a rational function $\frac{P(x)}{Q(x)}$ is not proper, then we can perform the long division so that we have $\frac{P(x)}{Q(x)} = (\text{polynomial}) + \frac{R(x)}{Q(x)}$ where $\frac{R(x)}{Q(x)}$ is proper

---

Evaluate $\int \frac{x^3 + x}{x - 1}\,dx$ first performing long division

Sol: Example 1, pg 533

---

Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$. first performing partial fraction decomposition

Sol: Example 2, pg 534

---

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$ first performing long division and then applying partial fraction decomposition

Sol: Example 4, pg 536

---

Evaluate $\int \frac{\sqrt{x + 4}}{x}\,dx$ first turning the problem into a problem of integrating a rational function (using substitution $u = \sqrt{x+4}$)

Example 7, pg 540

# 7.5 part I

Helpful fact given: $\dfrac{6t}{(3t-5)(3t+5)} = \dfrac{1}{3t-5} + \dfrac{1}{3t+5}$ by partial fraction decomposition.
(you don't need to verify)

<u>Question</u> Describe all possible Calc 2 methods for solving $\int \dfrac{6x}{9x^2-25}\,dx$ and describe the first few steps.

---

☑ The very first step is to perform partial fraction decomposition

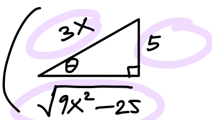
$\boxed{\dfrac{6x}{9x^2-25} = \dfrac{1}{3x-5} + \dfrac{1}{3x+5}}$
(given above - you don't need to compute)

. Then use the fact that $\int \dfrac{1}{3x+b}\,dx = \boxed{\dfrac{\ln|3x+b|}{3}} + C$

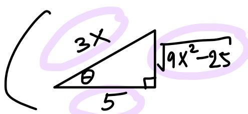
---

☑ Use u-substitution with $u = \boxed{9x^2-25}$, $du = \boxed{18x\,dx}$

---

☑ Use trig substitution with (

so $\dfrac{3x}{5} = \boxed{\csc\theta}$) OR

I prefer → (

so $\dfrac{3x}{5} = \boxed{\sec\theta}$)

---

☐ Perform Rationalizing Substitution with $u = \boxed{\phantom{xx}}$, $dx\ \boxed{\phantom{xx}}$

to get $\boxed{\int \qquad du}$ then apply partial fraction decomposition.

NO: "Rationalizing substitution" is to turn a non-rational integrand into a rational integrand using substitution.
But $\dfrac{6x}{9x^2-25}$ is already a rational function, $\dfrac{\text{polynomial}}{\text{polynomial}}$

# 7.5 part II

Helpful fact given: $\frac{1}{(t-1)t} = \frac{1}{t-1} - \frac{1}{t}$ by partial fraction decomposition. (you don't need to verify)

Question: Select all possible methods for solving $\int \frac{1}{1+e^x}dx$ and describe the first few steps. Leave unhelpful methods unchecked.

---

☐ The very first step is to perform partial fraction decomposition $\boxed{\phantom{xx}}$

Do not select

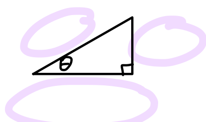
Partial fraction decomposition only applies to rational functions.
To use partial fraction method on a non-rational function,
you must first apply Rationalizing Substitution (end of Sec 7.4)

---

☑ Multiply $\frac{1}{1+e^x}$ by $\frac{e^{-x}}{e^{-x}}$, then do u-substitution $u = \boxed{e^{-x}+1}$, $du = \boxed{-e^{-x}\,dx}$

$$\int \frac{-e^{-x}}{e^{-x}+1}dx = \int \frac{-1}{u}du = -\ln|u| + C = -\ln\left(e^{-x}+1\right) + C$$

---

☐ Use trig substitution with  so $x = \boxed{\phantom{xx}}$

Do not select

Trig substitution is only helpful when Pythagorean thm can be applied,
i.e. when the integrand contains a factor in the form $\sqrt{\boxed{\phantom{x}}^2 + \boxed{\phantom{x}}^2}$
like $\int \frac{dx}{(\text{quadratic polynomial})^n}$ or $\int \frac{1}{x^2\sqrt{x^2-16}}dx$

---

☑ Perform Rationalizing Substitution with $u = \boxed{1+e^x}$, $\boxed{dx = \frac{1}{u-1}}$

$u-1 = e^x$
$\ln(u-1) = x$

$$\int \frac{1}{u}\frac{1}{u-1}du = \int\left(\frac{1}{u-1} - \frac{1}{u}\right)du = \ln|u-1| - \ln|u| + C$$
$$= \ln|e^x| - \ln|1+e^x| + C$$
$$= x - \ln|1+e^x| + C$$

(given above - you don't need to compute)

# 7.8 part I

1. Select all improper integrals.
Leave choices which are not improper integrals unchecked

☑ $\int_3^7 \frac{1}{\sqrt{x-3}}\,dx$

Improper integral because $\frac{1}{\sqrt{x-3}}$ has infinite discontinuity at a point in $[3,7]$: $\lim_{x\to 3^+} \frac{1}{\sqrt{x-3}} = \infty$

☐ $\int_3^7 \frac{e^x}{x+3}\,dx$

Don't select

The function $\frac{e^x}{x+3}$ is continuous at every point in $[3,7]$

☑ $\int_6^\infty \frac{1}{x^2+3}\,dx$

The region $[6,\infty)$ is unbounded, so this is an improper integral

☑ $\int_1^\infty \frac{1}{x^2}\,dx$

The region $[1,\infty)$ is unbounded, so this is an improper integral

---

2. Suppose we know that

If $t > 2$, then $\int_2^t f(x)\,dx = 1 - \frac{\ln(t)+1}{t}$. Evaluate $\int_2^\infty f(x)\,dx$.

Answer: $\int_2^\infty f(x)\,dx \overset{def}{=} \lim_{t\to\infty} \int_1^t f(x)\,dx$

$= \lim_{t\to\infty} 1 - \frac{\ln(t)+1}{t}$

$= 1 - \lim_{t\to\infty} \frac{\ln(t)+1}{t}$

$= 1 - \lim_{t\to\infty} \frac{\left(\frac{1}{t}\right)}{1}$ (by l'Hospital's Rule $\frac{\infty}{\infty}$)

$= 1 - 0$

# 7.8 part II

**3** Evaluate $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$.

Answer: $\int_1^\infty \frac{1}{x}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| - \ln(1) = \infty$

$\int_1^\infty \frac{1}{x^2}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$

---

**4** If $\int_t^5 f(x)\,dx = 2\left(\sqrt{3} - \sqrt{t-2}\right)$ for $t$ in $(2,3]$, (i) evaluate $\int_2^5 f(x)\,dx$.

(ii) Is $\int_2^5 f(x)\,dx$ convergent or divergent? (ii) $\int_2^5 f(x)\,dx$ is convergent.

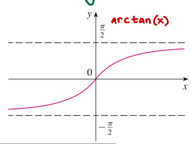
Answer: (i) $\int_2^5 f(x)\,dx \overset{\text{def}}{=} \lim_{t\to 2^+} \int_t^5 f(x)\,dx = \lim_{t\to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) = 2\sqrt{3}$

---

**5** If $\int_0^t f(x)\,dx = \arctan(t)$ for $t > 0$, (i) evaluate $\int_0^\infty f(x)\,dx$.

(ii) Is $\int_0^\infty f(x)\,dx$ convergent or divergent? (ii) $\int_0^\infty f(x)\,dx$ is convergent.

Answer: (i) $\int_0^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{t\to\infty} \int_0^t f(x)\,dx = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$




---

**6** If $\int_0^t f(x)\,dx = \ln|t-1|$ for $t$ in $[0,1)$, (i) evaluate $\int_0^1 f(x)\,dx$.

(ii) Is $\int_0^1 f(x)\,dx$ convergent or divergent? (ii) $\int_0^1 f(x)\,dx$ is divergent

Answer: (i) $\int_0^1 f(x)\,dx \overset{\text{DEF}}{=} \lim_{t\to 1^-} \int_0^t f(x)\,dx = \lim_{t\to 1^-} \ln|t-1| = -\infty$

---

**7** If $\int_0^t f(x)\,dx = \sin(t)$ for $t$ in $[0,\infty)$, (i) evaluate $\int_0^\infty f(x)\,dx$.

(ii) Is $\int_0^\infty f(x)\,dx$ convergent or divergent? (ii) $\int_0^\infty f(x)\,dx$ is divergent.

Answer: (i) $\int_0^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{t\to\infty} \int_0^t f(x)\,dx = \lim_{t\to\infty} \sin(t)$ does not exist.

# Sec 7.8 Review part III Answer Key

some helpful facts (you don't need to check):

ⓘ $\int_1^\infty \frac{1}{x}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| - \ln(1) = \infty$

ⓘⓘ $\int_1^\infty \frac{1}{x^2}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$

ⓘⓘⓘ $e^{-x}$ is positive for all $x$ in $[1,\infty)$ (in fact, for all $x$)

Question Use either $\int_1^\infty \frac{1}{x}\,dx$ or $\int_1^\infty \frac{1}{x^2}\,dx$ to show using the Comparison Test that $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent / convergent (choose one).

Answer

- Since $e^{-x}$ is always positive, we have $0 \le \frac{1}{x} \le \frac{1+e^{-x}}{x}$ for $x \geq 1$
- $\int_1^\infty \frac{1}{x}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| - \ln(1) = \infty$
- By the Comparison Thm, $\int_1^\infty \frac{1+e^{-x}}{x}\,dx = \infty$
- we have shown that $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.

— the end of answer —

## Strategies

How did I know to try using $\int_1^\infty \frac{1}{x}\,dx$ instead of $\int_1^\infty \frac{1}{x^2}\,dx$?

1. I see that $\frac{1+e^{-x}}{x}$ is more similar to $\frac{1}{x}$ (than to $\frac{1}{x^2}$)
2. I see that $\frac{1+e^{-x}}{x} = \frac{1}{x} + \frac{e^{-x}}{x}$ so $\frac{1+e^{-x}}{x} > \frac{1}{x}$ for $x$ in $[1,\infty)$.
3. It's also true that $\frac{1+e^{-x}}{x} > \frac{1}{x^2}$ for $x$ in $[1,\infty)$, but it's not helpful since $\int_1^\infty \frac{1}{x^2}\,dx$ converges

# Sec 7.8 Review part IV Answer key

some helpful facts (you don't need to check):

(i) $\int_1^\infty \frac{1}{x}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| - \ln(1) = \infty$

(ii) $\int_1^\infty \frac{1}{x^2}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$

(iii) Since $0 < \arctan(x) < \frac{\pi}{2}$ for $x$ in $[1,\infty)$, we have $0 < \frac{1}{\pi}\arctan(x) < \frac{1}{2} < 1$ for $x$ in $[1,\infty)$.

Question: Use either $\int_1^\infty \frac{1}{x}\,dx$ or $\int_1^\infty \frac{1}{x^2}\,dx$ to show using the Comparison Test that $\int_1^\infty \frac{\arctan(x)}{\pi\, x^2}\,dx$ is divergence / convergent (choose one).

Answer

- Since $0 < \arctan(x) < \frac{\pi}{2}$ for $x$ in $[1,\infty)$, we have $0 < \frac{1}{\pi}\arctan(x) < \frac{1}{2} < 1$ for $x$ in $[1,\infty)$.
- So $0 < \frac{1}{\pi}\frac{\arctan(x)}{x^2} < \frac{1}{x^2}$ for $x$ in $[1,\infty)$
- $\int_1^\infty \frac{1}{x^2}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$
- In particular, $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent
- By the Comparison Thm, $\int_1^\infty \frac{1}{\pi}\frac{\arctan(x)}{x^2}\,dx$ also is convergent.

— the end of answer (where we show that $\int_1^\infty \frac{1}{\pi}\frac{\arctan(x)}{x^2}\,dx$ is convergent) —

## Strategies

How did I know to try using $\int_1^\infty \frac{1}{x^2}\,dx$ instead of $\int_1^\infty \frac{1}{x}\,dx$?

1. I see that $\frac{\arctan(x)}{\pi\, x^2}$ is more similar to $\frac{1}{x^2}$ (than to $\frac{1}{x}$)
2. The hint says $0 < \frac{\arctan(x)}{\pi} < 1$ so $\frac{\arctan(x)}{\pi\, x^2} < \frac{1}{x^2}$ for $x$ in $[1,\infty)$.
3. It's also true that $\frac{\arctan(x)}{\pi\, x^2} < \frac{1}{x}$ for $x$ in $[1,\infty)$,

but it's not helpful since $\int_1^\infty \frac{1}{x}\,dx = \infty$.

WARNING: $\int_1^\infty \frac{1}{\pi}\frac{\arctan(x)}{x^2}$ is equal to a number, but it's not equal to $\int_1^\infty \frac{1}{x^2}\,dx = 1$.