Math 2924 Sec 7.1 – 7.5 Review
Answer key

1

Strategy:
Seeing $\sqrt{\text{polynomial}}$, try Rationalizing substitution Sec 7.4

$$\int x^2\sqrt{5x+8}\,dx = \int \underbrace{\left[\frac{u^2-8}{5}\right]^2}_{2}\underbrace{u}_{1}\underbrace{\frac{2u}{5}\,du}_{3}$$

1 $\boxed{u=\sqrt{5x+8}}$

$$u^2 = 5x+8$$
$$u^2-8 = 5x$$
$$\frac{u^2-8}{5} = x$$

2 $\boxed{\left[\frac{u^2-8}{5}\right]^2 = x^2}$

3 $\boxed{dx = \frac{2u}{5}\,du}$

$$\int x^2\sqrt{5x+8}\,dx = \int \left[\frac{u^2-8}{5}\right]^2 u\,\frac{2u}{5}\,du$$

$$= \int \frac{1}{25}\left[u^4-16u^2+64\right]u\,\frac{2}{5}u\,du$$

$$= \frac{2}{125}\int \left[u^6-16u^4+64u^2\right]du$$

$$= \frac{2}{125}\left[\frac{u^7}{7}-16\frac{u^5}{5}+64\frac{u^3}{3}\right]+C$$

$$= \boxed{\frac{2}{125}\left[\frac{(5x+8)^{\frac{7}{2}}}{7}-\frac{16}{5}(5x+8)^{\frac{5}{2}}+\frac{64}{3}(5x+8)^{\frac{3}{2}}\right]+C}$$

#2

Strategy:

- Seeing $\sqrt{\text{polynomial}}$ try sub $w = \sqrt{\text{polynomial}}$
- Seeing product of exponential function and a polynomial, try integration by parts

$$\int_1^4 e^{\sqrt{x}}\,dx = \int_{w=1}^{w=\sqrt{4}=2} 2e^w w\,dw$$

$$\left[\begin{array}{l} w = \sqrt{x} \\ w^2 = x \\ dx = 2w\,dw \end{array}\right]$$

$u = w$	$dv = e^w\,dw$
$du = dw$	$v = e^w$

$$= 2\left[we^w\Big|_1^2 - \int_1^2 e^w\,dw\right]$$

$$= 2\left[we^w - e^w\right]\Big|_{w=1}^{w=2}$$

$$= 2\left(\left[2e^2 - e^2\right] - \left[e - e\right]\right)$$

$$= \boxed{4e^2 - 2e^2}$$

(First sub then IBP 7.1)

Similar to Sec 7.1 Exercise 37

37. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Thus, $\int e^{\sqrt{x}}\,dx = \int e^t(2t)\,dt$. Now use parts with $u = t$, $dv = e^t\,dt$, $du = dt$, and $v = e^t$ to get $2\int te^t\,dt = 2t\,e^t - 2\int e^t\,dt = 2t\,e^t - 2e^t + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$.

#3

$$\int \sin(\sqrt{x})\, dx = \int \sin(u)\, 2u\, du$$

Strategy:

- Seeing $\sqrt{\text{polynomial}}$ try sub $u = \sqrt{\text{polynomial}}$
- Seeing product of trig function and a polynomial, try integration by parts

$\boxed{u = \sqrt{x}}$

$u^2 = x$

$\boxed{2u\, du = dx}$

(Alternatively, do $du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$du = \frac{1}{2}\frac{1}{u}\,dx$

$2u\, du = dx$)

you can use w instead of u if you want to be consistent with the previous problem

$$2\int \sin(u)\, u\, du = 2\left[u(-\cos(u)) - \int -\cos u\, du\right]$$

$U = u$	$dv = \sin(u)\,du$
$dU = du$	$v = -\cos(u)$

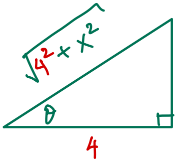
$$= 2\left[-u\cos u + \sin u\right] + C$$

$$= \boxed{2\left[-\sqrt{x}\cos(\sqrt{x}) + \sin(\sqrt{x})\right] + C}$$

#4

$$\int_0^2 \frac{1}{\sqrt{16+x^2}}\,dx$$

hint: $\frac{d}{dx} \ln\left|\sec(x)+\tan(x)\right| = \sec(x)$

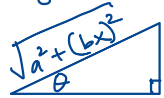


- $\frac{x}{4} = \tan\theta$ where θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $x = 4\tan\theta$
- $\boxed{dx = 4(\sec\theta)^2\, d\theta}$
- $\frac{4}{\sqrt{4^2+x^2}} = \cos\theta$
- $\boxed{\frac{1}{\sqrt{4^2+x^2}} = \frac{1}{4}\cos\theta}$

$x=2 \Rightarrow \frac{1}{2} = \tan\theta \Rightarrow \theta = \arctan\left(\frac{1}{2}\right)$

$x=0 \Rightarrow 0 = \tan\theta \Rightarrow \theta = \arctan(0)$

Strategy: Seeing $\left(a^2+(bx)^2\right)^n$, do trig substitution with



$$\int_0^{\arctan(\frac{1}{2})} \cos\theta\,(\sec\theta)^2\,d\theta = \int_0^{\arctan(\frac{1}{2})} \sec\theta\,d\theta$$

$$= \ln\left|\sec x + \tan x\right| \Big|_0^{\arctan(\frac{1}{2})}$$

$$= \ln\left|\sec\left(\arctan\left(\tfrac{1}{2}\right)\right) + \tan\left(\arctan\left(\tfrac{1}{2}\right)\right)\right| - \ln\left|\sec(0)+\tan(0)\right|$$

$$= \ln\left|\frac{\sqrt{5}}{2}+\frac{1}{2}\right| - \ln|1+0|$$

$$= \boxed{\ln\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)}$$

Recall: from Sec 7.2

$$\int \sec x\,dx = \int \sec x\,\frac{\sec x+\tan x}{\sec x+\tan x}\,dx$$

$$= \int \frac{(\sec x)^2 + \sec x\tan x}{\sec x+\tan x}\,dx$$

$u = \sec x + \tan x$

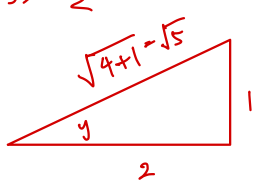
$du = \sec x\tan x + (\sec x)^2\,dx$

$$= \int \frac{du}{u}$$

$$= \ln\left|\sec x + \tan x\right| + C$$

Let $y = \arctan\left(\frac{1}{2}\right) \Rightarrow \tan(y) = \frac{1}{2}$

$$\sec(y) = \frac{1}{\cos y} = \frac{\sqrt{5}}{2}$$



#5

$$\int_1^{\sqrt{3}} [redacted] \arctan(1/x)\, dx$$

Integration by parts

$u = \arctan\left(\frac{1}{x}\right)$ $\quad dv = dx$

$du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)$ $\quad v = x$

Strategy: Seeing inverse functions of familiar functions like ln, arccos, arsin, try Integration by parts with dv=dx

$$uv\Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} v\, du = x \arctan\left(\tfrac{1}{x}\right)\Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} -\frac{1}{x}\,\frac{1}{1+\left(\frac{1}{x}\right)^2}\, dx$$

$$\left(\frac{1}{x+\frac{1}{x}} = \frac{1}{\left(\frac{x^2+1}{x}\right)} = \frac{x}{x^2+1}\right)$$

$$= x \arctan\left(\tfrac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\, dx$$

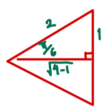
$U = x^2+1$

$dU = 2x\, dx$

$\frac{1}{2}\, dU = x\, dx$

$$= x \arctan\left(\tfrac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_{U=2}^{U=4} \frac{1}{2}\,\frac{1}{U}\, dU$$

$$= \sqrt{3}\,\underbrace{\arctan\left(\tfrac{1}{\sqrt{3}}\right)}_{\frac{\pi}{6}} - 1\,\underbrace{\arctan(1)}_{\frac{\pi}{4}} + \frac{1}{2}\left[\ln 4 - \ln 2\right]$$



$$= \boxed{\sqrt{3}\,\frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\left[\ln 4 - \ln 2\right]}$$

From Sec 7.1 Exercise 30

30. Let $u = \arctan(1/x)$, $dv = dx$ $\Rightarrow$ $du = \dfrac{1}{1+(1/x)^2}\cdot\dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2+1}$, $v = x$. By (6),

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = \left[x \arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x\,dx}{x^2+1} = \sqrt{3}\,\frac{\pi}{6} - 1\cdot\frac{\pi}{4} + \frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

#6

u-sub
+
trig sub (7.3)
+
trig identities (7.2)

$$\int x\sqrt{1-x^4}\,dx = \frac{1}{2}\int\sqrt{1-u^2}\,du$$

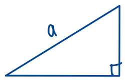
$u = x^2$

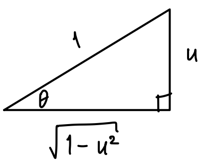
$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

strategy:

- Because you see $1-x^4$ and x, try u-substitution
- Seeing $\sqrt{a^2-(bu)^2}$, do trig substitution with





Let $\frac{u}{1} = \sin\theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\boxed{du = \cos\theta\,d\theta}$$

$$\boxed{\frac{\sqrt{1-u^2}}{1} = \cos\theta}$$

$$\int x\sqrt{1-x^4}\,dx = \frac{1}{2}\int\sqrt{1-u^2}\,du$$

$$= \frac{1}{2}\int(\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int\frac{1}{2}\left[1+\cos(2\theta)\right]d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$\sin x\cos x = \frac{1}{2}\sin(2x)$

(will be given on exams if needed)

$$= \frac{1}{4}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$

$$= \frac{1}{4}\left(\arcsin(u) + u\sqrt{1-u^2}\right) + C$$

$$= \boxed{\frac{1}{4}\left[\arcsin(x^2) + x^2\sqrt{1-x^4}\right] + C}$$

#7

$$\int x^3 [\cos(x^4)]^3 [\sin(x^4)]^2 \, dx = \frac{1}{4}\int [\cos(u)]^3 [\sin(u)]^2 du$$

Let $u = x^4$

$du = 4x^3 \, dx$

$\frac{1}{4} du = x^3 \, dx$

Strategy:

- Seeing x^4 and x^3, try u-substitution
- Seeing $[\cos(u)]^n [\sin(u)]^m$,

know that strategies in Sec 7.2 will work.

Since $\cos(u)$ has odd power, save one factor of $\cos(u)$ and apply $(\cos(u))^2 = 1-(\sin(u))^2$

Save one $\cos(u)$ from $[\cos(u)]^3$

$$\frac{1}{4}\int [\cos(u)]^2 [\sin(u)]^2 \cos(u) \, du = \frac{1}{4}\int [1-(\sin(u))^2] [\sin(u)]^2 \cos u \, du$$

Let $w = \sin u$

$dw = \cos u \, du$

$$= \frac{1}{4}\int (1-w^2) w^2 \, dw$$

$$= \frac{1}{4}\int [w^2 - w^4] \, dw$$

$$= \frac{1}{4}\left[\frac{w^3}{3} - \frac{w^5}{5}\right] + C$$

$$= \boxed{\frac{1}{4}\left[\frac{[\sin(x^4)]^3}{3} - \frac{[\sin(x^4)]^5}{5}\right] + C}$$

#8

$$\int x^2 \ln(1+x)\,dx = \frac{x^3}{3}\ln(1+x) - \int \frac{1}{3}\frac{x^3}{1+x}\,dx$$

Strategy:

- Seeing a product of a polynomial and ln, try integration by parts
- The result is a rational function $\frac{\text{a polynomial}}{\text{another polynomial}}$ Any rational function can be integrated using techniques from Sec 7.4.

$$\left[\begin{array}{ll} u = \ln(1+x) & dv = x^2\,dx \\ du = \frac{1}{1+x}\,dx & v = \frac{x^3}{3} \end{array}\right]$$

$$\int \frac{x^3}{1+x}\,dx = \int \left[x^2 - x + 1 - \frac{1}{1+x}\right] dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|1+x| + C$$

$$\begin{array}{r} x^2 - x + 1 \\ 1+x \,\overline{)\, x^3 \qquad} \\ x^3 + x^2 \\ \hline -x^2 \\ -x^2 - x \\ \hline x \\ x + 1 \\ \hline -1 \end{array}$$

$$\int x^2 \ln(1+x)\,dx = \frac{x^3}{3}\ln(1+x) - \int \frac{1}{3}\frac{x^3}{1+x}\,dx$$

$$= \boxed{\frac{x^3}{3}\ln(1+x) - \frac{1}{3}\left[\frac{x^3}{3} - \frac{x^2}{2} + x - \ln|1+x|\right] + C}$$