Name: ____________________________________

Math 1152Q: Fall 2018
Week 8 Sample Quiz

Sec 11.3 Integral Test, p-series and Estimates of Sum

1. (Concept) Suppose f is a continuous, positive, and decreasing function on $[1, \infty)$. Suppose $a_k = f(k)$ for $k = 1, 2, 3, \ldots$.

   (a) Draw pictures for illustrating the quantities of each of the following.

$$\int_1^6 f(x)\ \mathrm{dx} \qquad \sum_{k=2}^{6} a_k \qquad \sum_{k=1}^{5} a_k$$

   (b) Then rank the three quantities in increasing order.

   (c) What are the conditions needed to apply the Integral Test ?

Solution: On pages 1-2 of `https://egunawan.github.io/fall18/lecture_notes_f18/f18_week7_day2_notes11_3part1.pdf`. See also Sec 11.3 Exercise 2 page 725

2. (Integral Test) Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

   a. Explain why the integral test can be applied.

   Solution: Let $f(x) = \dfrac{1}{x(\ln x)^5}$. Then $f(x)$ is continuous and positive on for $x \geq 2$. It is also decreasing on $[2, \infty)$ since $x(\ln x)^5$ is a product of increasing functions on $[2, \infty)$. Thus we can use the integral test.

   b. Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.

   Solution: Use u-substitution $u = \ln(x)$.

   c. Evaluate $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^5}$.

   Solution: Use u-substitution $u = \ln(x)$.

   d. Apply the integral test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

   Solution: Since $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^5}$ converges by part (c), we conclude by the integral test that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ also converges.

3. (Integral test and estimates of sums) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$.

   a. Verify that the integral test *can* be used to decide if this series converges.

Solution:
Let $f(x) = x/3^x$. We will show that $f(x)$ is positive, continuous, and decreasing on $[1, \infty)$.

This function is positive and continuous for $x \geq 1$ since x is a polynomial, 3^x is an exponential function, and 3^x is never 0 on $[1, \infty)$.

To show that $f(x)$ is decreasing on $[1, \infty)$, compute $f'(x) = \dfrac{1 - x \ln 3}{3^x}$, which implies $f'(x) < 0$ for $x > \dfrac{1}{\ln 3}$. Since $\ln 3 > \ln e = 1$, we have $\dfrac{1}{\ln 3} < 1$, so $f'(x) < 0$ for $x \leq 1$. This justifies the use of the integral test on $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$.

b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.

Solution: compute $\displaystyle\int \frac{x}{3^x}\, dx$, try integration by parts:
either $\boxed{u = x \text{ and } dv = dx/3^x = 3^{-x}\, dx}$ or $\boxed{u = dx/3^x = 3^{-x} \text{ and } dv = xdx}$.
You should get

$$\int_1^{\infty} \frac{x}{3^x}\, dx \quad = \quad \lim_{b \to \infty} \left(-\frac{b}{3^b \ln 3} - \frac{1}{3^b (\ln 3)^2} + \frac{1}{3 \ln 3} + \frac{1}{3 (\ln 3)^2} \right).$$

Since $\displaystyle\lim_{b \to \infty} b/3^b = 0$ by L'Hospital's Rule and $\displaystyle\lim_{n \to \infty} 1/3^b = 0$,

$$\int_1^{\infty} \frac{x}{3^x}\, dx = \frac{1}{3 \ln 3} + \frac{1}{3 (\ln 3)^2}.$$

Since $\displaystyle\int_1^{\infty} \frac{x}{3^x}\, dx$ converges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$ also converges by the Integral Test.

c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

Solution: The Nth remainder R_N is at most $\displaystyle\int_N^{\infty} \frac{x}{3^x}\, dx = \lim_{b \to \infty} \int_N^b \frac{x}{3^x}\, dx = \frac{N \ln 3 + 1}{3^N (\ln 3)^2}$.
See Week 8 day 1 notes
`https://egunawan.github.io/fall18/lecture_notes_f18/f18_week8_day1_hw11_3.pdf`

d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

Solution: See Week 8 day 1 notes
`https://egunawan.github.io/fall18/lecture_notes_f18/f18_week8_day1_hw11_3.pdf`

4. (Integral Test from 11.3 WebAssign)

(a) Find the values of p for which the integral $\displaystyle\int_e^{\infty} \frac{6}{x (\ln x)^p}$ dx converges. Evaluate the integral for these values of p.
(Hint: Your work should look like Example 4 on pg 530. Check the three cases, when $p = 1$, when $p < 1$, and when $p > 1$.)

Solution: Answer: $\boxed{p > 1 \text{ converges. Otherwise, diverges.}}$

(b) Evaluate the integral $\displaystyle\int_1^\infty \frac{3}{x^6}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^\infty \frac{3}{n^6}$ is convergent or divergent.

Solution: Answer: $= \boxed{3/5}$

(c) Evaluate the integral $\displaystyle\int_1^\infty \frac{1}{(4x+2)^3}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^\infty \frac{1}{(4n+2)^3}$ is convergent or divergent.

Solution: Answer: $= \boxed{1/288}$

(d) Evaluate the integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x+9}}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^\infty \frac{1}{\sqrt{n+9}}$ is convergent or divergent.

Solution: Answer: $\boxed{\text{divergent}}$

(e) Evaluate the integral

$$\int_1^\infty x\ e^{-9x}\ \text{dx}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^\infty \frac{n}{e^{9n}}$ is convergent or divergent.

Solution: Answer: $\boxed{10/81e^9}$

(f) The following statement is false: "If $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\displaystyle\int_1^\infty f(x)\,dx$ converges then $\displaystyle\sum_{n=1}^\infty a_n = \int_1^\infty f(x)\,dx$."
Give a counterexample by coming up with a continuous, positive, and decreasing $f(x)$ on $[1,\infty)$ and computing both $\displaystyle\sum_{n=1}^\infty a_n$ (where $a_n := f(n)$) and $\int_1^\infty f(x)\,dx$, showing that they are not equal.
(Hint: you know how to compute precisely the sum of any convergent geometric series).

Solution: Counterexample: Let $f(x) = \frac{1}{2^x}$. The sum of the series $\displaystyle\sum_{n=0}^\infty \frac{1}{2^n}$ is $\frac{1}{1-\frac{1}{2}} = 2$ (think the infinite mathematicians joke), so $\displaystyle\sum_{n=1}^\infty \frac{1}{2^n} = 2 - 1 = 1$, while $\displaystyle\int_1^\infty \frac{dx}{2^x} = \lim_{b\to\infty}\left(\frac{1}{2^x(-\ln 2)}\Big|_1^b\right) = \frac{1}{2\ln 2} \neq 1.$

5. (Section 11.3 True/False)

(a) For each statement, determine whether it's true or false and give a brief justification:

i. The series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ converges by the p-series test.

ii. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ diverges by the p-series test.

iii. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

iv. The exact sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is an open question.

v. $\int_5^{\infty} \frac{1}{x^2}\,\mathrm{dx} = \frac{1}{5}$

vi. $\int_5^{\infty} \frac{1}{x^2}\,\mathrm{dx} = 5$

(Answers: see Sec 11.3, page 722)

(b) Is the following statement true or false? Justify.
Suppose $f(x)$ is a continuous function defined on $[5,\infty)$. If $f(x)$ is not bounded on $[5,\infty)$, we cannot apply the integral test using $\int_5^{\infty} f(x)\,\mathrm{dx}$. .

Solution: Answer to 5(b) is True. Justification: If $g(x)$ is positive and decreasing on $[5,\infty)$, then it is bounded (for example, bounded below by 0 and bounded above by $g(5)$). The contrapositive of this statement is: If $g(x)$ is not bounded, then it is not positive or not decreasing. Since $g(x)$ does not meet at least one of the criteria for applying the integral test using $\int_5^{\infty} g(x)\,\mathrm{dx}$, we cannot apply the integral test using $\int_5^{\infty} g(x)\,\mathrm{dx}$.

6. (Sec 11.3 p-series and shifting indices)
(Note: the symbol ζ is the lower-case Greek letter which is pronounced "zeta" in English).
The *Riemann zeta*-function ζ is defined by

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

It is used in number theory to study the distribution of prime numbers.

(a) What is the domain of the function ζ? (That is, for what values of x is this function defined?)
(Hint: go to Sec 11.3, page 722)

Solution: Answer: The domain of the function is the set of x such that the series is convergent, that is, when $x > 1$, so the domain of ζ is $(1,\infty)$.

(b) Euler computed $\zeta(2)$ to be $\frac{\pi^2}{6}$. (See page 720, sec 11.3). Use this fact to find the sum of each series below.
Hint: Given a convergent series, you can multiply out a constant, and subtract terms as needed.

$$\sum_{n=3}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(5n)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+3)^2}$$

Solution: $\frac{\pi^2}{6} - \frac{1}{2^2} - 1 = \frac{\pi^2}{6} - \frac{5}{2}$ and
$\frac{1}{25}\frac{\pi^2}{6}$ and
$\frac{\pi^2}{6} - 1$ and
$\frac{\pi^2}{6} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = \pi^2/6 - \frac{49}{36}$