Name: ______________________________

Math 1152Q: Fall 2018
Week 8 Sample Quiz

Sec 11.3 Integral Test, p-series and Estimates of Sum

1. (Concept) Suppose f is a continuous, positive, and decreasing function on $[1, \infty)$. Suppose $a_k = f(k)$ for $k = 1, 2, 3, \ldots$.

   (a) Draw pictures for illustrating the quantities of each of the following.

   $$\int_1^6 f(x)\ \mathrm{dx} \qquad \sum_{k=2}^{6} a_k \qquad \sum_{k=1}^{5} a_k$$

   (b) Then rank the three quantities in increasing order.

   (c) What are the conditions needed to apply the Integral Test ?

2. (Integral Test) Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

   a. Explain why the integral test can be applied.

   b. Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.

   c. Evaluate $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^5}$.

   d. Apply the integral test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

3. (Integral test and estimates of sums) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$.

   a. Verify that the integral test *can* be used to decide if this series converges.

   b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.

   c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

   d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

4. (Integral Test from 11.3 WebAssign)

   (a) Find the values of p for which the integral $\displaystyle\int_e^{\infty} \frac{6}{x(\ln x)^p}\ \mathrm{dx}$ converges. Evaluate the integral for these values of p.
   (Hint: Your work should look like Example 4 on pg 530. Check the three cases, when $p = 1$, when $p < 1$, and when $p > 1$.)

   (b) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{3}{x^6}\ \mathrm{dx}$. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_{1}^{\infty} \frac{3}{n^6}$ is convergent or divergent.

   (c) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{1}{(4x+2)^3}\ \mathrm{dx}$. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_{1}^{\infty} \frac{1}{(4n+2)^3}$ is convergent or divergent.

   (d) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x+9}}\ \mathrm{dx}$. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_{1}^{\infty} \frac{1}{\sqrt{n+9}}$ is convergent or divergent.

(e) Evaluate the integral

$$\int_1^{\infty} x\, e^{-9x}\ \mathrm{dx}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^{\infty} \frac{n}{e^{9n}}$ is convergent or divergent.

(f) The following statement is false: "If $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)\,dx$."
Give a counterexample by coming up with a continuous, positive, and decreasing $f(x)$ on $[1, \infty)$ and computing both $\sum_{n=1}^{\infty} a_n$ (where $a_n := f(n)$) and $\int_1^{\infty} f(x)\,dx$, showing that they are not equal.
(Hint: you know how to compute precisely the sum of any convergent geometric series).

5. (Section 11.3 True/False)
   (a) For each statement, determine whether it's true or false and give a brief justification:
      i. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test.
      ii. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ diverges by the p-series test.
      iii. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
      iv. The exact sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is an open question.
      v. $\int_5^{\infty} \frac{1}{x^2}\ \mathrm{dx} = \frac{1}{5}$
      vi. $\int_5^{\infty} \frac{1}{x^2}\ \mathrm{dx} = 5$

   (Answers: see Sec 11.3, page 722)
   (b) Is the following statement true or false? Justify.
   Suppose $f(x)$ is a continuous function defined on $[5, \infty)$. If $f(x)$ is not bounded on $[5, \infty)$, we cannot apply the integral test using $\int_5^{\infty} f(x)\ \mathrm{dx}$. .

6. (Sec 11.3 p-series and shifting indices)
   (Note: the symbol ζ is the lower-case Greek letter which is pronounced "zeta" in English).
   The *Riemann zeta*-function ζ is defined by

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

   It is used in number theory to study the distribution of prime numbers.
   (a) What is the domain of the function ζ? (That is, for what values of x is this function defined?)
   (Hint: go to Sec 11.3, page 722)
   (b) Euler computed $\zeta(2)$ to be $\frac{\pi^2}{6}$. (See page 720, sec 11.3). Use this fact to find the sum of each series below.
   Hint: Given a convergent series, you can multiply out a constant, and subtract terms as needed.

$$\sum_{n=3}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(5n)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+3)^2}$$