**Name:** ______________________________

**Math 1152Q: Fall 2018**
**Week 7 Quiz sample**

Summary:

- Integration by parts
- apply partial fraction decomposition to rational function (with denominators that have deg 2 or 3);
- determine whether an improper integral diverges or converges; evaluate the convergent integral;

1. (Optional Extra Credit) Write 10 or more distinct (correctly spelled) preferred (first) names of students who are present during the quiz (whose name is not the same as your name) and be able to match names to faces.

   Example: 'write the first names of the students who are currently sitting in the back row'.

## 1 Sec 7.1 only, integration by parts

2. (Integration by parts)

   1.) Evaluate $\int \ln\left(x + \sqrt{1+x^2}\right)$ dx or 2.) $\int x \ \tan^2 x$ dx or 3.) $\int \cos(\sqrt{x})$ dx or

   4.) Evaluate $\int x^2 (\ln x)^2$ dx or 5.) omitted or 6.) $\int \cos(\ln x)$ dx or

   7a.) How can you derive the formula for Integration by Parts? or

   7b.) Evaluate $\int_0^{\frac{\pi}{2}} x \ \cos(2x)$ dx or

   7c.) Suppose $f(1)$=2, $f(4)$=7, $f'(1) = 5$, $f'(4)$=3. Suppose $f''$ is continuous. Evaluate $\int_1^4 x \ f''(x)$ dx

   or 7d.) Evaluate $\int \arctan x$ dx or 7e.) $\int e^x \cos x$ dx or

   7f.) A particle that moves along a straight line has velocity $v(t) = t^3 e^{-t}$ meters per second after $t$ seconds. How far will it travel during the first $t$ seconds?

   **Solution:** `https://egunawan.github.io/fall18/notes/hw7_1key.pdf`

3. (a) Evaluate $\int (x+2)\sin(3x)$ dx

   **Solution:** Answer: Integration by parts with $\boxed{u = x+2 \text{ and } dv = \sin(3x) \text{ dx}}$.

   Then $\int (x+2)\sin(3x)$ dx = $\boxed{-\frac{1}{3}(x+2)\cos(3x) + \frac{1}{9}\sin(3x) + \text{Constant}}$.

   (b) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

## 2 Sec 7.4 only, integrals of rational functions

4. (Integrating rational functions)

(a) Evaluate $\int \frac{2x+1}{x^2-4}\,dx$. Although it is possible to evaluate this using the trig substitution method, please evaluate by first doing partial fraction decomposition method.

**Solution:** Note: It is also possible to evaluate this using the trig substitution method, but the following will walk you through the Partial Fraction Decomposition method.

*Thinking about the problem:*

The integrand $\frac{2x+1}{x^2-4}$ is a rational function and does not look like it can be handled with substitution, so we use partial fractions. The denominator $x^2-4$ is $(x+2)(x-2)$, a product of different linear factors, so the partial fraction decomposition of $\frac{2x+1}{x^2-4}$ is $\frac{A}{x+2}+\frac{B}{x-2}$ for some constants $A$ and $B$. After solving for $A$ and $B$ we would have $\int \frac{2x+1}{x^2-4}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x-2}\,dx$ and can integrate the right side.

*Doing the Problem:*

Writing $\frac{2x+1}{x^2-4} = \frac{A}{x+2}+\frac{B}{x-2}$, solve for $A$ and $B$ by multiplying both sides by the denominator $x^2-4$:

$$2x+1 = A(x-2)+B(x+2).$$

Setting $x=2$, we find

$$2(2)+1 = 5 = A(0)+B(2+2) = 4B \Rightarrow 5 = 4B \Rightarrow B = \frac{5}{4}.$$

Setting $x=-2$, we find

$$2(-2)+1 = A(-2-2)+B(0) \Rightarrow -3 = -4A \Rightarrow A = \frac{3}{4}.$$

Therefore $\frac{2x+1}{x^2-4} = \frac{3/4}{x+2}+\frac{5/4}{x-2}$, so

$$\int \frac{2x+1}{x^2-4}\,dx = \frac{3}{4}\int \frac{dx}{x+2} + \frac{5}{4}\int \frac{dx}{x-2}$$
$$= \boxed{\frac{3}{4}\ln|x+2| + \frac{5}{4}\ln|x-2| + C}.$$

(b) Evaluate $\int \frac{10}{(x+5)(x-2)}$ dx.

**Solution:** Answer: $\boxed{\frac{10}{7}\left(\ln(x-2)-\ln(5+x)\right)} + C$

(c) Provide a computation that is either a formal verification (that is, a proof) or simply a sanity-check for your answer to the previous question.

**Solution:** For example, you can check that the derivative of your result is equal to $\frac{10}{(x+5)(x-2)}$.

(d) Evaluate $\int \frac{9}{(x-6)(x+3)}$ dx or $\int \frac{12}{(x-2)(x+1)}$ dx or $\int \frac{8}{(x-1)(x+3)}$ dx.

**Solution:** Answer: After applying partial fraction decomposition,
the integrand is equal to $\frac{1}{x-6}-\frac{1}{x+3}$ or $\frac{4}{x-2}-\frac{4}{x+1}$ or $\frac{2}{x-1}-\frac{2}{x+3}$.

The antiderivative is $\boxed{\ln\left|\frac{x-6}{x+3}\right| + C}$ or $\boxed{4\ln\left|\frac{x-2}{x+1}\right| + C}$ or $\boxed{2\ln\left|\frac{x-1}{x+3}\right| + C}$.

(e) Evaluate $\int \frac{x+4}{x^2+2x+5}$ dx

## 3 7.8 Proper and improper Integrals and evaluating them using Sec 7.1, 7.4, and u-substitution

5. (Sec 7.8 improper integrals)

(a) Find the values of $p$ for which the integral $\int_e^\infty \frac{6}{x(\ln x)^p}$ dx converges. Evaluate the integral for these values of $p$.
(Hint: u-substitution. Check what happens when $p = 1$, when $p < 1$, and when $p > 1$.)

(b) Determine whether

$$\int_2^\infty \left(\frac{1}{e^5}\right)^x \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.

**Solution:** $1/(5\ 10^e)$

(c) Determine whether

$$\int_2^\infty \frac{1}{x^2 + 8x - 9} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.

**Solution:** You can use partial fraction decomposition. ln(11)/10

(d) Determine whether

$$\int_0^1 \frac{4}{x^5} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.

**Solution:** Answer: divergent

(e) Determine whether

$$\int_0^1 \frac{4}{x^{0.5}} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.

**Solution:** 8

(f) Determine whether

$$\int_2^3 \frac{2}{\sqrt{3-x}} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.

**Solution:** 4

(g) Evaluate the integral

$$\int_1^\infty \frac{3}{x^6} \text{ dx.}$$

**Solution:** 3/5

(h) Evaluate the integral

$$\int_1^\infty \frac{1}{(4x+2)^3}\,\text{dx}.$$

**Solution:** 1/288

(i) Evaluate the integral

$$\int_1^\infty \frac{1}{\sqrt{x+9}}\,\text{dx}.$$

**Solution:** divergent

(j) Evaluate the integral

$$\int_1^\infty x\ e^{-9x}\ \text{dx}.$$

**Solution:** $10/81e^9$

(k) Evaluate the integral

$$\int_1^\infty x\ e^{-9x}\ \text{dx}$$

**Solution:** $10/81e^9$

6. (Various definite and indefinite integrals).

(a) Is $\int_1^2 t^3 \ln(t)dt$ a proper or improper integral?

**Solution:** $\int_1^2 t^3 \ln(t)dt$ is a $\boxed{\text{proper}}$ integral since neither $t^3$ nor $\ln(t)$ has discontinuity on the interval $[1,2]$.

(b) Evaluate $\int t^3 \ln(t)dt$ and $\int_1^2 t^3 \ln(t)dt$

**Solution:** Answer: $\boxed{\frac{1}{4}\left(t^4 \ln(t) - \frac{t^4}{4}\right) + C}$ and $\boxed{\ln(16) - \frac{15}{16}}$.

(c) Perform either a proof or a reality check for the previous problem. For example, differentiate your answer (for a proof) or check that your definite integral is a positive number, since $t^3 ln(t)$ is positive for $t > 1$.

(d) Is $\int_0^1 x\ e^{-x^2}$ a proper or improper integral?

**Solution:** $\int_0^1 x\ e^{-x^2}$ is a $\boxed{\text{proper}}$ integral: Both $-x^2$ and $e^x$ is continuous on $[0,1]$, so $e^{-x^2}$ is also continuous on $[0,1]$, since the composition of continuous functions is continuous (reference: Sec 2.5 Theorem 9, pg 121). Since $x$ is continuous on $[0,1]$ and the product of continuous functions is continuous (reference: Sec 2.5 Theorem 4, pg 117), we conclude that $x\ e^{-x^2}$ is continuous on $[0,1]$ (and, in fact, everywhere - but this fact isn't relevant to this situation).

(e) Evaluate $\int_0^1 x\ e^{-x^2}$

**Solution:** Use u-sub with $\boxed{u=-x^2}$, $du=-2x$ dx. Then $\int_0^1 x\ e^{-x^2} = -\frac{1}{2}e^u\Big|_0^{-1} = \boxed{\frac{1}{2}(1-e^{-1})}$.

(f) Perform a reality check for your answer to the previous problem.

**Solution:** A possible reality check: your answer should be positive because $xe^{-x^2}$ is positive on the interval $(0,1]$.

(g) Evaluate $\int e^{\sqrt{x}}$ dx

**Solution:** Answer: Do substitution with $w=\sqrt{x}$ and $dw=\frac{1}{2}x^{-\frac{1}{2}}$ dx. Then do integration by parts with $u=w$ and $dv=e^w$ dx. Then $\int e^{\sqrt{x}}\text{ dx} = 2we^w - 2\int e^w dw = 2we^w - 2e^w =$ $\boxed{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + \text{C}}$.

(h) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(i) Is $\int_0^9 \frac{1}{\sqrt{x}}$ dx an improper integral?

**Solution:** Yes, $\lim_{x\to 0^+}\frac{1}{\sqrt{x}}=\infty$, so $\frac{1}{\sqrt{x}}$ has an infinite discontinuity at 0.

(j) Determine whether $\int_0^{25}\frac{1}{\sqrt{x}}$ dx or $\int_0^{16}\frac{1}{\sqrt{x}}$ dx or $\int_0^{9}\frac{1}{\sqrt{x}}$ dx is convergent or divergent. If it is convergent, evaluate the integral.

**Solution:** $\boxed{5*2}$ or $\boxed{4*2}$ or $\boxed{3*2}$.

7. (From class handouts)

(a) Is the integral $\int_0^\pi \sin^3(5x)\,dx$ a proper or improper integral?

**Solution:** This is a proper integral because $\sin(5x)$ is continuous on $[0,\pi]$, and so $(\sin(5x))^3$ is also continuous on $[0,\pi]$.

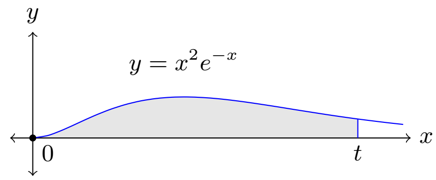
(b) Without explicitly trying to evaluate this integral, determine whether it is possible for $\int_0^\infty x^2e^{-x}\,dx$ to be convergent to a negative value.

**Solution:** It is not possible because $x^2e^{-x} \geq 0$ for all $x \geq 0$. See sketch in the solution of the next part.

(c) Is $\int_0^\infty x^2e^{-x}\,dx$ convergent or divergent? If convergent, evaluate it.

**Solution:** *Thinking about the problem:*

The integral is $\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$ and the graph of $y = x^2e^{-x}$ is below. We will compute $\int_0^t x^2e^{-x}\,dx$ and see how it behaves as $t \to \infty$.




To evaluate $\int_0^t x^2e^{-x}\,dx$ we will use integration by parts.

*Doing the problem:*

To work out $\int_0^t x^2e^{-x}\,dx$ with integration by parts set $u$ and $dv$ to be as in the chart below, and then compute $du$ and $v$.

| | |
|---|---|
| $u = x^2$ | $dv = e^{-x}\,dx$ |
| $du = 2x\,dx$ | $v = -e^{-x}$ |

Thus $\int_0^t x^2e^{-x}\,dx = uv\Big|_0^t - \int_0^t v\,du = -x^2e^{-x}\Big|_0^t + \int_0^t 2xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx$. We work out the new integral also using integration by parts, starting with the chart below.

| | |
|---|---|
| $u = x$ | $dv = e^{-x}\,dx$ |
| $du = dx$ | $v = -e^{-x}$ |

Thus

$$\int_0^t xe^{-x}\,dx = -xe^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx = -\frac{t}{e^t} - e^{-x}\Big|_0^t = -\frac{t}{e^t} - \frac{1}{e^t} + 1,$$

so returning to the initial calculation we have

$$\int_0^t x^2e^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\left(-\frac{t}{e^t} - \frac{1}{e^t} + 1\right) = -\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2.$$

Letting $t \to \infty$, $\int_0^\infty x^2e^{-x}\,dx = \lim_{t\to\infty}\left(-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2\right) = 0 - 0 - 0 + 2$ by L'Hospital's rule (used twice for the first expression). Thus $\int_0^\infty x^2e^{-x}\,dx = 2$: the improper integral is convergent and equals 2.

(d) Evaluate 1.) $\int_0^\infty e^{-2x}\,\mathrm{dx}$ or 2.) $\int_1^\infty \frac{1}{\sqrt{x}}$ or 3.) $\int_1^\infty \sin^2 x\,\mathrm{dx}$ or 4.) $\int_1^\infty \frac{1}{x^2+2x-3}\,\mathrm{dx}$.

**Solution:** `https://egunawan.github.io/fall18/notes/LA7_8part1key.pdf`

(e) 1.) Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}}\,\mathrm{dx}$ or 2.) Evaluate $\int_1^{\frac{\pi}{2}} \sec x$ or 3.) Evaluate $\int_0^5 \frac{x}{x-2}\,\mathrm{dx}$ or

4.) Use the Comparison Theorem to determine whether $\int_1^\infty \frac{x}{x^3+1}\,\mathrm{dx}$ converges.

**Solution:** `https://egunawan.github.io/fall18/notes/LA7_8part2key.pdf`

8. (Sec 7.8 Improper integrals)

(a) When is an integral improper? Hint: There are two kinds. Copy definitions from Sec 7.8 pg 527 and 531.

(b) When is an integral proper? (The answer is when it is not improper, but explain what needs to happen for a definite integral to be proper).

(c) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^{\infty} f(x)\ \mathrm{dx} = \lim_{t\to\infty} \int_a^t f(x)\ \mathrm{dx}$ **T** **F**

**Justification:**

**Solution:** Answer: $\boxed{\text{True}}$, by definition

(d) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^{\infty} f(x)\ \mathrm{dx} = \lim_{a\to\infty} \int_a^t f(x)\ \mathrm{dx}$ **T** **F**

**Justification:**

**Solution:** Answer: $\boxed{\text{False}}$, by definition

(e) True or False, and why? The integral $\int_2^3 \sqrt{x-2}\ \mathrm{dx}$ is improper. **T** **F**

**Solution:** Answer: $\boxed{False}$. The function $\sqrt{x-2}$ is continuous on $[2,3]$.

(f) True or False, and why? $\int_0^1 \frac{27}{x^5}\ \mathrm{dx}$ is improper. **T** **F**

**Solution:** Answer: $\boxed{True}$. The function $\frac{27}{x^5}$ has an infinite discontinuity at 0 since $\lim_{x\to 0} \frac{27}{x^5}$ does not exist. $\lim_{x\to 0^+} \frac{27}{x^5} = \infty$

(g) Evaluate $\int_0^1 \frac{27}{x^5}\ \mathrm{dx}$

**Solution:** Answer: $\lim_{a\to 0^+} \int_a^1 \frac{27}{x^5} = \infty$. $\boxed{\text{The integral diverges}}$.

(h) True or False? $\int_{-1}^1 \frac{1}{x}\ \mathrm{dx}$ is improper. **T** **F**

**Solution:** Answer: $\boxed{True}$. The function $\frac{1}{x}$ has an infinite discontinuity at 0 since $\lim_{x\to 0} \frac{1}{x}$ does not exist. $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

(i) Evaluate $\int_{-1}^1 \frac{1}{x}\ \mathrm{dx}$

**Solution:** $\int_0^1 \frac{1}{x}\ \mathrm{dx} = \lim_{a\to 0^+} \int_a^1 \frac{1}{x} = \infty$. $\boxed{\text{The integral diverges}}$.

(j) Determine whether $\int_0^1 9x^2 \ln(x)$ dx converges or diverges. If it converges, evaluate it.

**Solution:** Answer: -1

(k) Evaluate $\int_4^8 \frac{4}{x\sqrt{x^2-16}}dx$ OR $\int_{-7}^7 \frac{1}{\sqrt{49-x^2}}dx$. If it converges, evaluate it.

**Solution:** $\int_4^8 \frac{4}{x\sqrt{x^2-16}}dx$ ans:$\pi/3$, $\int_{-7}^7 \frac{1}{\sqrt{49-x^2}}dx$. ans:$\pi$

(l) Determine whether $\int_e^\infty \frac{1}{x(\ln x)^3}$ dx converges or diverges. If it converges, evaluate the integral.

**Solution:** Answer: 1/2

(m) Determine whether

$$\int_2^\infty \left(\frac{1}{e^5}\right)^x \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.

**Solution:** Answer: $\boxed{1/(5\ 10^e)}$

(n) Determine whether $\int_2^\infty \frac{1}{x^2+8x-9}$ dx is convergent or divergent. If it is convergent, evaluate it.

**Solution:** Use partial fraction decomposition (probably faster) or complete the square + trig substitution (probably longer). Answer: $\boxed{\ln(11)/10}$.

(o) Determine whether $\int_0^1 \frac{4}{x^5}$ dx is convergent or divergent. If it is convergent, evaluate it.

**Solution:** Answer: $\boxed{\text{divergent}}$

(p) Determine whether $\int_0^1 \frac{4}{x^{0.5}}$ dx is convergent or divergent. If it is convergent, evaluate it.

**Solution:** Answer: $\boxed{8}$

(q) Determine whether $\int_2^3 \frac{2}{\sqrt{3-x}}$ dx is convergent or divergent. If it is convergent, evaluate it.

**Solution:** Answer: $\boxed{4}$

(r) Write 2 improper integrals (different from above) so that one is convergent and the other is divergent.

(s) Write 2 proper (definite) integrals that are different from above.

(t) Write 2 indefinite integrals.