9.3 Separable Equations

Separable equation. A first-order differential equation is called *separable* if it has the form $\frac{dy}{dx} = g(x)f(y)$. The right side is a product of separate functions of x and of y.

Example. Are any of

a.) $\frac{dy}{dx} + y = e^x$, b.) $\frac{dy}{dx} = x + y$, c.) $(y^2 + xy^2)\frac{dy}{dx} = 1$, d.) $\frac{dy}{dx} = 2y\left(1 - \frac{y}{10000}\right)$

separable differential equations? Why?

Practice Sec 9.3 Examples 1,2,3,4 in Stewart textbook.

Orthogonal trajectory. An *orthogonal trajectory* of a family of curves is a curve intersecting each curve of the family at right angles (orthogonally).

Recall that a line perpendicular to a line with slope m has slope ________. So, if a family of curves has derivative $\frac{dy}{dx} = m(x, y)$, then the derivative of an orthogonal trajectory to that family of curves is $\frac{dy}{dx} =$ ________________.

Applications.

Example. Consider the family of circles centered at the origin with radius K: in Cartesian coordinates, ______________________ and, in polar coordinates, ___________. Find *an* orthogonal trajectory of this family of curves. Find all the orthogonal trajectories of this family.

Optional HW: Copy from Stewart Sec 9.3 Example 5). The curves $x = ky^2$ are a family of parabolas. Find *an* orthogonal trajectory of this family of curves. Find all the orthogonal trajectories.

Mixing problems. If $y(t)$ denotes the amount of substance in a tank at time t, then its rate of change is $\frac{dy}{dt} = \text{(rate in)} - \text{(rate out)}$.

Applications.

Mixing Problem Example 1: A tank contains 1000L of pure water. Brine that contains .005 kg of salt per liter of water enters the tank at a rate of 5L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15L/min. How much salt is in the tanks (a) after t minutes, (b) after one hour, and (c) in the long run?

Mixing Problem Example 2: A tank contains 500 L of brine with 15 kg of dissolved salt. Brine having .2 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and is drained from the tank at 10 L/min. How much salt is in the tank after t minutes? After 20 min? In the long run?

Thinking about the problem:

Let $y(t)$ be the amount of salt in the tank at t min. We need the rate in and rate out of salt in kg/min. The rate in is the concentration of salt (in kg/L) multiplied by the rate of liquid entering the tank (in L/min), and the rate out is the concentration of salt multiplied by the rate of liquid leaving the tank. After finding $\frac{dy}{dt}$, we solve for $y(t)$ and $y(20)$.

Doing the problem: Let $y(t)$ be the amount of kg of salt in the tank at t minutes, so $y(0) = 15$. The problem also says brine with .2 kg/L of salt enters at a rate of 10 L/min and the whole mixture drains from the tank at 10 L/min. The concentration of salt entering the tank at time t is .2 kg/L, so the rate of salt entering the tank at time t is

$$\text{concentration} \cdot \text{rate of liquid entering the tank} = .2\,\frac{\text{kg}}{\text{L}} \cdot 10\,\frac{\text{L}}{\text{min}} = 2\,\frac{\text{kg}}{\text{min}}.$$

The concentration of salt leaving the tank at time t is

$$\frac{\text{amount of salt in tank}}{\text{volume of tank}} = \frac{y(t)\,\text{kg}}{500\,\text{L}},$$

so the rate of salt leaving of the tank at time t is

$$\text{concentration} \cdot \text{rate of liquid leaving the tank} = \frac{y(t)\text{kg}}{500\,\text{L}} \cdot 10\,\frac{\text{L}}{\text{min}} = \frac{y(t)}{50}\,\frac{\text{kg}}{\text{min}}.$$

Therefore, in kg/min,

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = 2 - \frac{y(t)}{50} = \frac{100 - y(t)}{50}$$

The differential equation $\boxed{\frac{dy}{dt} = \frac{100-y}{50}}$ is separable:

$$\frac{dy}{dt} = \frac{100-y}{50} \Longrightarrow \frac{dy}{100-y} = \frac{dt}{50} \Longrightarrow \int \frac{dy}{100-y} = \int \frac{dt}{50} \Longrightarrow -\ln|100-y| = \frac{t}{50} + C.$$

Thus $\ln|100 - y| = -t/50 - C$, so raising e to both sides, we get $100 - y(t) = \pm e^{-C}e^{-t/50}$. Setting $t = 0$ here, $100 - 15 = \pm e^{-C}$, so $100 - y(t) = 85e^{-t/50}$. Thus $y(t) = 100 - 85e^{-t/50}$. This is the number of kilograms of salt in the tank after t minutes. After 20 minutes there is $y(20) = 100 - 85e^{-20/50} \approx 43.02$ kg of salt.

Remark. In the long run, the concentration of salt in the tank must match that of incoming brine (.2 kg/L), so the amount of salt in 500 L should tend to (.2 kg/L)(500 L)= 100 kg, which is consistent with $y(t) \to 100$ as $t \to \infty$.

Solutions should show all of your work, not just a single final answer.

1. Find the solution of $\frac{dy}{dx} = e^x e^y$ where $y(0) = 1$.
2. Find the general solution of the differential equation $(y^2 + xy^2)y' = 1$.
3. Find the orthogonal trajectories of the family of curves $y^4 = kx^3$, where k is constant.

4. (From WebAssign) A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

   (a) Let $y(t)$ be the amount of alcohol in the vat after t minutes. What is $y(0)$?

   (b) What is the rate the alcohol is entering the vat?

   (c) What is the rate the alcohol is leaving the vat?

   (d) What is $\frac{dy}{dt}$?

   (e) Solve the separable equation from (d).

   (f) Find the percentage of alcohol after one hour (what should t be?).

5. A tank contains 1000 L of pure water. Brine that contains .5 kg of salt per liter of water enters the tank at a rate of 20 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 20 L/min. How many kg of salt are in the tank after t minutes? After one hour (round to one digit after the decimal point)?

6. True/False (with justification)

   The differential equation $\frac{dy}{dx} = yx + y$ is separable.

   The differential equation $\frac{dy}{dx}y = e^{4x}$ is separable.

   The differential equation $\frac{dy}{dx} + e^{y} = e^{x}$ is separable.

   The differential equation $\frac{dy}{dx}e^{5y} = e^{x}$ is separable.