NAME: ________________________ Notes

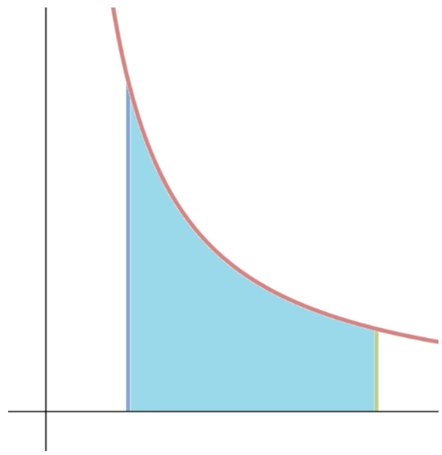
Recall

The **improper integral** is used for cases in which

- The **interval** of integration is **infinite** or
- The **integrand** has an **infinite discontinuity** on the interval of integration.

Infinite Discontinuity

Consider the integral $\int_c^1 \frac{1}{\sqrt{x}}\,dx$, where $0 < c < 1$.



$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = \left(2\sqrt{x}\right)_c^1 = 2 - 2\sqrt{c}$$

$c = \frac{1}{4}$	$c = \frac{1}{9}$	$c = \frac{1}{16}$		$c \to 0^+$ or
$2 - 2\sqrt{\frac{1}{4}}$	$2 - 2\sqrt{\frac{1}{9}}$	$2 - 2\sqrt{\frac{1}{16}}$		or

We express this result as

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

which is an improper integral because 0 leads to a zero-denominator.

Definition **Type 2: Improper Integrals with an Unbounded Integrand (copy from pg 531)**

1. If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

or _________________________________,

provided this limit exists (as a finite number).

2. If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_,$$

or _________________________________,

provided this limit exists (as a finite number).

The improper integrals $\int_a^b f(x)\,dx$ is called

- **convergent** if the corresponding limit exists and
- **divergent** if the limit does not exist.

3. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Example: Sketch and evaluate $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$. (Hint: first evaluate the indefinite integral using trig substitution)

Example (copy Example 8 pg 532: Use integration by parts and l'Hopital's Rule):

a) Sketch and evaluate $\int_0^1 \ln x \, dx$. (Explain your work involving L'Hospital.)

b) Evaluate $\lim_{t \to 0^+} t\,(\ln t)^2$ or $\lim_{t \to 0^+} t\,(\ln t)^3$ (Apply L'Hospital multiple times.)

c) Evaluate $\int_0^1 (\ln x)^2 \; dx$ or $\int_0^1 (\ln x)^3 \; dx$. Use the previous part (b) to help you get your answer.

d) Evaluate $\int_0^1 (\ln x)^n \; dx$ for n = 2, 3, 4, 5, 6 using a computing tool (like WolframAlpha). Predict the value for any n.

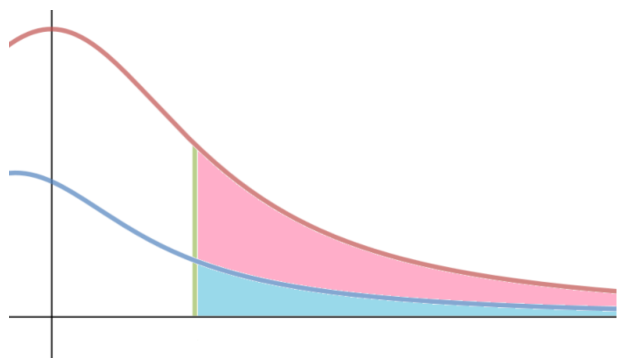
A Comparison Test for Improper Integrals

Theorem **Comparison Theorem (copy from pg 533)**

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)\, dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is ______________________________.

2. If $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is ______________________________.



Caution

- If $\int_a^\infty g(x)\, dx$ is convergent, then ______________________________.
- If $\int_a^\infty f(x)\, dx$ is divergent, then ______________________________.

Example 10 (copy from pg 534):

Use the comparison test to show that the integral $\int_1^\infty \frac{1+e^{-x}}{x}\, dx$ is convergent/divergent.