

7.3 Trigonometric Substitution

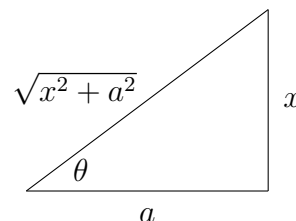
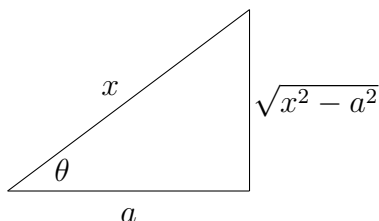
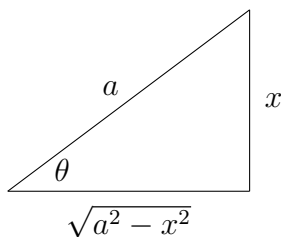
In each of the following trigonometric substitution problems, draw a triangle and label an angle and all three sides corresponding to the trigonometric substitution you select.

Summary of Trigonometric Substitution (understand how to make the triangles).

$$\sqrt{a^2 - x^2}$$

$$\sqrt{x^2 - a^2}$$

$$\sqrt{x^2 + a^2}$$



$$\sin \theta = \frac{x}{a}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

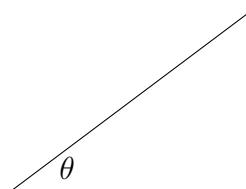
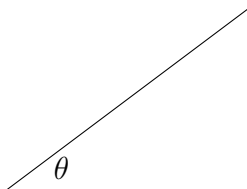
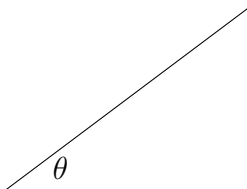
$$\cos \theta = \frac{a}{x}, \text{ that is, } \underline{\hspace{2cm}}$$

$$\tan \theta = \frac{x}{a}, \text{ where}$$

OR

OR

OR



$$\underline{\hspace{2cm}} = \frac{x}{a}, \text{ where}$$

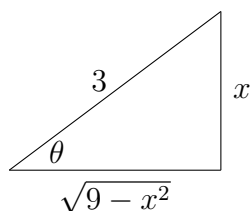
$$\underline{\hspace{2cm}} = \frac{a}{x}, \text{ that is, } \underline{\hspace{2cm}}$$

$$\underline{\hspace{2cm}} = \frac{x}{a}, \text{ where}$$

Example A (indefinite integral): Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$.

Thinking about the problem:

Since the integrand involves $\sqrt{9-x^2}$ and there is not an extra factor of x in the numerator (if there were it might be possible to do a u -substitution with $u = 9-x^2$), we will try a trigonometric substitution corresponding to a right triangle with a leg of length $\sqrt{9-x^2}$, hypotenuse 3, and the other leg has length x .



Doing the problem: Using the triangle diagram above,

$$\sin \theta = \frac{x}{3} \quad \text{where} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

so $x = 3 \sin \theta$. Then $dx = 3 \cos \theta d\theta$. Also from the triangle $\cos \theta = \frac{\sqrt{9-x^2}}{3}$, so $\sqrt{9-x^2} = 3 \cos \theta$. The integral becomes

$$\begin{aligned} \int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{3 \cos \theta d\theta}{3 \cos \theta} \\ &= \int d\theta \\ &= \theta + C. \end{aligned}$$

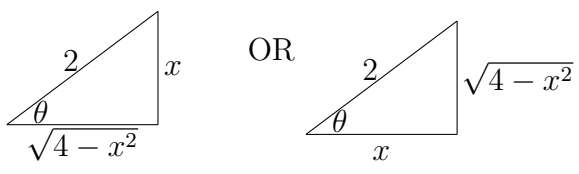
Since the substitution we used was $x = 3 \sin \theta$, $\theta = \arcsin\left(\frac{x}{3}\right)$. So

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin\left(\frac{x}{3}\right) + C.$$

Example B (definite integral): Evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} dx$.

Thinking about the problem: Since the integrand involves $\sqrt{4-x^2}$ and there is not just a factor of x in the numerator (otherwise we can do a u -substitution with $u = 4-x^2$), we try a trigonometric substitution with a triangle with a side of length $\sqrt{4-x^2}$, hypotenuse 2, and the other side has length x .

Doing the problem:

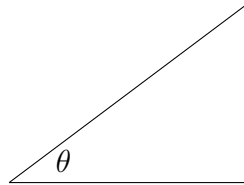


OR

Solutions should show all of your work, not just a single final answer.

1. Evaluate $\int \frac{dx}{(9+x^2)^{3/2}}$.

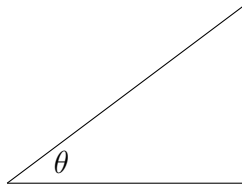
- (a) Fill in the sides of the right triangle below where $\sqrt{9+x^2}$ is one of the sides.



- (b) Using the sides of the triangle in (a), compute the indefinite integral. Write the final answer in terms of x .

2. Evaluate $\int \frac{\sqrt{x^2 - 9}}{x^3} dx$.

- (a) Fill in the sides of the right triangle below where $\sqrt{x^2 - 9}$ is one of the sides.



- (b) Using the sides of the triangle in (a), compute the indefinite integral. Write the final answer in terms of x .

3. Evaluate the definite integral $\int_0^3 \frac{x^2}{\sqrt{9-x^2}} dx$. (*Tip:* When you make a trigonometric substitution, change the bounds of integration as part of the substitution. See Example B.)

4. T/F (with justification): To evaluate $\int \frac{dx}{x^2\sqrt{x^2+2}}$ by trigonometric substitution, use $x = 2 \tan \theta$.