

7.1 Integration by Parts

Integration By Parts. Reverse *product rule*

$$u v' + v u' = \frac{d}{dx}(uv) \quad ,$$

and put $v' = \frac{dv}{dx}$ and $u' = \frac{du}{dx}$ to get *integration by parts*.

For indefinite integrals,

$$\int u dv + \int v du = uv \quad ,$$

where u and v are both functions of x , so

$$\boxed{\int u dv = uv - \int v du .}$$

For definite integrals,

$$\int_a^b u dv + \int_a^b v du = uv \Big|_a^b \quad ,$$

so

$$\boxed{\int_a^b u dv = uv \Big|_a^b - \int_a^b v du .}$$

Example A: Evaluate $\int x^2 e^x dx$.

Thinking about the problem:

Since the integral is a product of “unrelated” functions x^2 and e^x , we will use integration by parts. We have to pick u and dv , and then du and v . Passing from u to du is differentiation, and for any u in practice we can get du . However, passing from dv to v involves integration and that can be tricky. *The key is to pick dv first:* let dv be the most “complicated” expression in the integrand that you already know how to integrate, and then u is just the rest of the integrand. That is more practical advice than acronym-based tricks you may read elsewhere.

Doing the problem:

To figure out $\int x^2 e^x dx$ with integration by parts, the function e^x is more complicated (more sophisticated) than x^2 . Set $dv = e^x dx$ and then $u = x^2$, so $u dv = x^2 e^x dx$

Next, we compute $du = 2x dx$ and $v = e^x$. Here are the results in a table.

$u = x^2$	$dv = e^x dx$
$du = 2x dx$	$v = e^x$

From $\int u dv = uv - \int v du$ we get

$$\int x^2 e^x dx = \int u dv = uv - \int v du = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2 \int x e^x dx.$$

Now we integrate by parts $\int x e^x dx$, which is simpler than what we started with. Again, e^x is the most complicated factor in the integrand that we know how to integrate. Set $dv = e^x dx$ and then $u = x$, so $u dv = x e^x dx$. We get $du = dx$ and $v = e^x$. as in the following table.

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

Then

$$\int x e^x dx = \int u dv = uv - \int v du = x e^x - \int e^x dx = x e^x - e^x + C.$$

Putting this all together,

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2(x e^x - e^x + C) \\ &= x^2 e^x - 2x e^x + 2e^x + 2C \\ &= x^2 e^x - 2x e^x + 2e^x + 2C. \end{aligned}$$

Since C is an arbitrary constant, we could change $2C$ to C in the last formula if we wished.

Example B: Evaluate $\int \tan^2 x \sec x \, dx$.

1. We want to evaluate $\int x e^{-3x} dx$.

(a) Fill in the following table. (Hint: pick dv first, then u .)

$u =$	$dv =$
$du =$	$v =$

(b) Evaluate the integral using integration by parts.

2. We want to evaluate $\int_0^\pi x \cos(3x) dx$.

(a) Fill in the following table. (Hint: pick dv first, then u .)

$u =$	$dv =$
$du =$	$v =$

(b) Evaluate $\int x \cos(3x) dx$ using integration by parts.

(c) Using part (b), evaluate $\int_0^\pi x \cos(3x) dx$.

3. Evaluate $\int_0^\pi x^2 \sin x dx$.

4. Evaluate $\int e^x \cos(3x) dx$.

5. In 'In-class work 7.2' https://egunawan.github.io/fall18/notes/notes7_2practice.pdf you found $\int \cos^2 x dx$ by the trick of writing $\cos^2 x$ in terms of $\cos(2x)$. Instead compute this with integration by parts. (Hint: $\sin^2 x + \cos^2 x = 1$.)

6. T/F (with justification): For differentiable $f(x)$, $\int_0^\pi f(x) \cos x dx = - \int_0^\pi f'(x) \sin x dx$.