11.9: Representations of Functions as Power Series

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Fall 2018

We've seen examples of convergent power series—but can we write an explicit function that is represented by a power series?

Consider $\sum_{n=0}^{\infty} x^n =$

This is geometric with ratio $r =$

The power series converges if

so the interval of convergence is

When it converges, the series converges to

Extending this idea

So for $|x| < 1$, we can express $\dfrac{1}{1-x}$ as a power series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Can we express $\dfrac{1}{1+x}$ as a power series? What values of x work?

Example

Find a power series representation for $f(x) = \dfrac{1}{3-x}$ and find its interval of convergence.

Question: What is the center? A. 0 B. 1 C. 2 D. 3 E. 4

Question: What is the radius of convergence? A. 1 B. 3 C. 1/3

Example

Find a power series representation for $f(x) = \dfrac{5}{1+4x^2}$ and find its interval of convergence.

Question: What is the radius of convergence?

A. 1 B. 2 C. 1/2 D. 4 E. 1/4

Find a power series representation for $f(x) = \dfrac{2x^4}{2-3x}$ and find its interval of convergence.

Example

What happens if we find an antiderivative for the equation below?

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

As it turns out, we can use both differentiation and integration to express other kinds of functions as powers series:

Theorem: If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(I) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(II) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence for both of these power series is R.

Example

Find a power series representation for $f(x) = \dfrac{1}{(5+x)^2}$ and find its interval of convergence.

Example

Use a power series to approximate $\int_0^{0.3} \ln\left(1+t^4\right)\,dt$ to six decimal places.

Find a power series representation for $f(x) = \dfrac{3}{8+7x}$ and find its interval of convergence.

Find $\displaystyle\int \frac{x^2}{1+8x^3}\,dx$ as a power series, and find its radius of convergence.

Find a power series representation for $f(x) = \dfrac{x}{(3+x)^2}$ and find its radius of convergence.

Find a power series representation for $f(x) = \dfrac{x}{(3+x)^3}$ and find its radius of convergence.