Power Series

A **power series** is a series for the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

The number a is called the **center** of the series.

Examples:

1. $\displaystyle\sum_{n=0}^{\infty} 2(x-1)^n = 2 + 2(x-1) + 2(x-1)^2 + \dots$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-4)^n}{n^2+2} = \frac{1}{2} + \frac{x-4}{3} + \frac{(x-4)^2}{11} + \dots$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{5^n} = 1 + \frac{x+2}{5} + \frac{(x+2)^2}{25} + \dots$

Convergence Example

Example: Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = 1 + \frac{x}{3} + \frac{x^2}{9} + \dots$$

where plugging in different values of x gives different series. For instance,

$x=-1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = 1 - \frac{1}{3} + \frac{1}{9} - \dots$

$x=4$: $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^n} = 1 + \frac{4}{3} + \frac{16}{9} + \dots$

What is the center of this series?

A. 0 **B.** 1 **C.** $\frac{1}{3}$ **D.** 3

For what interval of x-values does this power series converge (this is the **interval of convergence**)?

Theorem

For a given power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

there are only three possibilities

1. The series converges only when $x = a$
2. The series converges for all x
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. Note, when $|x-a| = R$, it could either converge or diverge. You must check these!

What are the intervals of convergence in each of these situations?

1. converges for all x in $[a, a]$
2. converges for all x in $(-\infty, \infty)$
3. converges for all $|x-a| < R$ i.e. for all x in $(a-R, a+R)$. This might not be the interval of convergence! We need to check $x = a-R$ and $x = a+R$ to see if either or both of these are also in the interval.

Radius of Convergence

The **radius of convergence**, usually denoted R, is the distance from the center to the of the series to one of the endpoints of the interval of convergence. For our three possible types of intervals of convergence, the radii of convergence are as follows:

1. interval $[a, a] \to$ radius $= 0$
2. interval $(-\infty, \infty) \to$ radius $= \infty$
3. interval $(a-R, a+R)$ (or similar) $\to$ radius $= R$

What is the radius of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = 1 + \frac{x}{3} + \frac{x^2}{9} + \dots$$

Example

Use ideas about geometric series to find the interval and radius of convergence for

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{5^n} = 1 + \frac{x+2}{5} + \frac{(x+2)^2}{25} + \dots$$

Ideas about geometric series work sometimes, but the Ratio Test will work in MANY more situations to help us determine the convergence of power series.

Example

Use the Ratio Test on the power series $\displaystyle\sum_{n=1}^{\infty} \frac{5^n (x-4)^n}{\sqrt{n}}$

Questions for previous Example

$$\sum_{n=1}^{\infty} \frac{5^n (x-4)^n}{\sqrt{n}} = 1 + (x-4) + \frac{(x-4)^2}{2} + \dots$$

What is the center of the series?

A. 0 **B.** 4 **C.** -4

What is the radius of convergence?

A. 5 **B.** 10 **C.** $\frac{1}{5}$ **D.** $\frac{2}{5}$

What is the interval of convergence?

Example

Find the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

What is the radius of convergence of this series?

A. 0 **B.** 1 **C.** 10 **D.** ∞

Example

Find the radius and interval of convergence for $\sum_{n=0}^{\infty} n!(x+8)^n$.

What is the radius of convergence of this series?

A. 0 **B.** 1 **C.** 8 **D.** ∞

Example

Find the radius and interval of convergence for $\sum_{n=0}^{\infty} 9^n (x-2)^{2n}$

What is the radius of convergence of this series?

A. 1 **B.** $\frac{1}{3}$ **C.** $\frac{1}{9}$ **D.** $\frac{1}{81}$

Example

Let $f(x) = 1 + 7x + x^2 + 7x^3 + x^4 + 7x^5 + \dots$
Find the interval of convergence of this series, and find an explicit formula for $f(x)$

More practice

Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{7^n n^2}$

Find the radius and interval of convergence of the power series
$$\sum_{n=0}^{\infty} \frac{10^n (x-2)^n}{4^n}$$

Find the radius and interval of convergence of the power series
$$\sum_{n=0}^{\infty} \frac{(-1)^n n! (x+3)^n}{11^n}$$

Find the radius and interval of convergence of the power series
$$\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{3^n n^5}$$

Find the radius and interval of convergence of the power series
$$\sum_{n=1}^{\infty} \frac{2^n (x-7)^{3n}}{n}$$