**Name: _______________** **Alternating Series** Notes

**TASK 1. Recall Strategy for series with positive terms** Fill in the blanks by looking up the answers from the given pages.

Determine whether the infinite series $\sum_{k=1}^{\infty} a_k$ with **positive terms** converge or diverge.

1. (11.2 page 710) **The Geometric Series** → when $\sum_{k=1}^{\infty} a_k$ has the form $\sum_{k=1}^{\infty} r^k$

   If $|r| \geq 1$, ____________________. If $|r| < 1$, ____________________.

2. (11.2 page 713) **The Divergence Test**

   If $\lim_{k\to\infty} a_k = 0$, ____________________. If $\lim_{k\to\infty} a_k \neq 0$, ____________________.

3. (11.2) **Harmonic Series, special case of *p*-Series** → $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ with p =1.

4. (11.2 Ex. 8) **The Telescoping Series** → when $\sum_{k=1}^{\infty} a_k$ can be reduced to $\sum_{k=1}^{\infty} (b_k - b_{k+1})$

   $S_n =$ ______________. If $\lim_{n\to\infty} S_n$ exists, ______________. Otherwise, ______________.

5. (11.4 page 727, optional) **The Comparison Test** → when none of the above methods works or when there is an obvious comparison.

   If $0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges, _______. If $0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ diverges, _______.

   If $0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ converges, _______. If $0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges, _______.

6. (11.4 page 729) **The Limit Comparison Test (most versatile)** → when $a_k$ involves dominant terms (p-series, rational function, or a geometric series). Consider $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.

   If $0 < L < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges, ________. If $0 < L < \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, ________.

   If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, ___________. If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ diverges, ___________.

   If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ converges, __________. If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, __________.

7. (11.4 page 728) **The *p*-Series** → when $\sum_{k=1}^{\infty} a_k$ has the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$

   If $p \leq 1$, ____________________. If $p > 1$, ____________________.

8. (11.6) **The Ratio Test** → when $a_k$ involves factorials or powers. Consider $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

   If $0 \leq r < 1$, ______________. If $r > 1$, ______________. If $r = 1$, ______________.

## Alternating Series

What if we are given a series $\sum(-1)^{n+1} b_n$ where $b_n > 0$? How do we determine whether the series converges?

**TASK 2.** Go to Sec 11.5 page 732-733. Read until just before the "proof of the alternating series test" (reading the proof is optional).

## Alternating Series Test

**Definition** **Alternating Series**

Suppose that $b_n > 0$ for all positive integer $n$, then the series

$$\sum_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots + (-1)^{n+1} b_n + \cdots$$

is called the **Alternating Series**.

Example (Example 1 pg 734): $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is called the **Alternating Harmonic Series**.

**TASK 3a.** Complete the following by copying from the reading.

**Theorem** **The Alternating Series Test**

The alternating series $\sum(-1)^{n+1} b_n$ converges provided

i.) $\{b_n\}$ is ____________________ for all $n$.

In other words, ______________________________.

ii.) $\lim_{n\to\infty} b_n =$ ____________________.

**TASK 3b.** **Procedure** (fill in the blanks)

If $\lim_{n\to\infty} b_n \neq 0$ and $\lim_{n\to\infty}(-1)^{n+1} b_n \neq 0$,

then $\sum(-1)^{n+1} b_n$ ____________________________________________________________

(by the ______________________________________________ Test from Sec 11.2 page 713).

If $\lim_{n\to\infty} b_n = 0$, check whether $\{b_n\}$ is decreasing.

- If $\{b_n\}$ is decreasing, then $\sum(-1)^{n+1} b_n$ ______________________________________

(by the ____________________________________________ Test, Sec 11.5 page 732).

**TASK 4 (**Example 1 page 734**):** Answer the following question by closely following the explanation for Example 1 page 734 in the book. It would be more effective if you first write your answer without looking at the book then check the book afterwards.

Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

**TASK 5.** Complete the following.

**Theorem Alternating Harmonic Series**

- The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ ______________________(see Sec 11.2 page 713 Example 9).
- The **alternating harmonic series** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ ________________(see your answer above).

**TASK 6.** Stay on page 734. Attempt to do Example 2 *AND* Example 3 on a separate paper without looking at the solution in the book. Check your solution with the book.

Please choose either Example 2 or Example 3 and write down its question and solution in the space below.

Do TASKS 7 and 8 by imitating Example 1, 2, and 3 (pg 734). Hint: one of them converges and the other diverges. Check your answers with WolframAlpha.

**TASK 7.**
Example:

Determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{5^n}$ converges.

Step (ii): Calculate $\lim_{n\to\infty} b_n =$

Step (i): Estimate (in your head or scratch paper) whether $\{b_n\}$ is decreasing. If you believe it is decreasing, verify by:

- citing a known fact from class (for example, behavior of an exponential function - Sec 11.1 Example 11 pg 700 or Sec 11.1 Eq. 4 pg 697),
- computing a derivative (follow Sec 11.5 Example 3 page 734),
- or directly (follow Sec 11.1 Example 13 page 701).

**TASK 8.**
Example:

Determine whether $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$ converges.

Step (ii): Calculate $\lim_{n\to\infty} b_n =$

Step (i): Estimate (in your head or scratch paper) whether $\{b_n\}$ is decreasing. If you believe it is decreasing, verify by:

- computing a derivative (follow Sec 11.5 Example 3 page 734),
- or directly (follow Sec 11.1 Example 13 page 701).