

## MATH 1152Q ANSWER TO WORKSHEET PROBLEM

Explanation for problem 2 from Sec 11.11 handout [https://egunawan.github.io/fall18/notes/notes11\\_11.pdf](https://egunawan.github.io/fall18/notes/notes11_11.pdf) problem 2.

### Problem 2:

Use Taylor's inequality to determine a partial sum for the Maclaurin series of  $f(x) = \cos x$  that is within 0.0001 of  $\cos 2$ .

### Answer:

Because the center of my Taylor polynomial will be  $a = 0$ , and I want to approximate  $\cos 2$ , I can set my interval of approximation to be  $|x - 0| \leq 2$ .

First, I find a positive number  $M$  so that  $|f^{(n)}(x)| \leq M$  for all  $|x| \leq 2$ . A number that works is  $\boxed{M = 1}$  because the  $n$ -th derivative of  $f(x)$  will be either  $\pm \sin x$  or  $\pm \cos x$ .

Next, by Taylor's inequality, we have

$$(\text{error for } T_N(x)) \leq \frac{M}{(N+1)!} |x|^{N+1} = \frac{|x|^{N+1}}{(N+1)!} \text{ for } |x| \leq 2$$

Therefore,

$$(\text{error for } T_N(2)) \leq \frac{|2|^{N+1}}{(N+1)!}$$

So my goal is to find a positive integer  $N$  so that

$$\boxed{\frac{2^{N+1}}{(N+1)!} \leq \frac{1}{10^4}}$$

**Possible Method 1:**

For example, if I choose  $N = 5$ , then

$$\begin{aligned} \frac{2^6}{6!} &= \frac{\cancel{2}\cancel{2}\cancel{2}.2.2.2}{6.5.\cancel{4}.3.\cancel{2}.1} \\ &= \frac{2^3}{6.3} \\ &> \frac{1}{18}, \text{ which is not smaller than } \frac{1}{10^4}, \text{ so we need } N \text{ bigger than } 5. \end{aligned}$$

If I choose  $N = 10$ , then

$$\begin{aligned} \frac{2^{11}}{11!} &= \frac{2.2.2.2.2.2.2.2.2.2.2}{11.10.9.8.7.6.5.4.3.2.1} \\ &= \frac{\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}}{11.10.9.\cancel{8}.7.6.5.\cancel{4}.3.\cancel{2}.1} \\ &= \frac{2.2.2.2.2}{11.10.9.7.6.5.3} \\ &= \frac{2.2.2}{11.5.9.7.3.5.3} \\ &= \frac{8}{11.(5.7.5).9.(3.3)} \\ &= \frac{8}{11.175.81} \\ &< \frac{1}{10^4}. \end{aligned}$$

So  $\boxed{N = 10}$  works, and  $T_{10}(x) = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - x^{10}/10!$  would give you an approximation within .0001 for  $\cos 2$ .

**Possible Method 2:** I see that

$$\begin{aligned} \frac{|2|^{N+1}}{(N+1)!} &= \frac{2}{(N+1)} \frac{2}{(N)} \frac{2}{(N-1)} \cdots \frac{2}{5} \frac{2}{4} \frac{2}{3} \frac{2}{2} \frac{2}{1} \\ &= 2 \left( \frac{2}{(N+1)} \frac{2}{(N)} \frac{2}{(N-1)} \cdots \frac{2}{5} \frac{2}{4} \frac{2}{3} \frac{2}{2} \right) \\ &< 2 \left( \frac{2}{N+1} 1.1 \dots 1.1.1.1 \right) \\ &= \frac{4}{N+1} \end{aligned}$$

Since I want

$$\frac{2^{N+1}}{(N+1)!} \leq \frac{1}{10^4},$$

it would be enough if

$$\frac{4}{N+1} \leq \frac{1}{10^4},$$

so it would be enough if

$$4(10^4) \leq N + 1.$$

So  $\boxed{N = 40000}$  also works, and  $T_{40000}(x) = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - x^{10}/10! + \cdots + x^{40000}/40000!$  would give you an approximation within .0001 (actually, much smaller) for  $\cos 2$ .