

Taylor series and Taylor polynomials of a function at a . If $f(x)$ can be written as a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ in an interval around a then c_n must be $\frac{f^{(n)}(a)}{n!}$. We call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

the *Taylor series* of $f(x)$ at a and we call the partial sum

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

the n th-degree *Taylor polynomial* of $f(x)$ at a . Ideally $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ for x near a , or equivalently $f(x) = \lim_{n \rightarrow \infty} T_n(x)$ for x near a , and to verify this in examples we can use Taylor's inequality below.

Maclaurin series. The Taylor series of $f(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

and is called the *Maclaurin series* of $f(x)$ ¹. Ideally $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for x near 0.

Taylor's inequality. A bound on the remainder $R_n(x) = f(x) - T_n(x)$, where $T_n(x)$ is a Taylor polynomial for $f(x)$ at a , is *Taylor's inequality*, which uses a bound on $|f^{(n+1)}(x)|$:

if $|f^{(n+1)}(x)| \leq M$ for all $|x-a| \leq d$, then $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$ if $|x-a| \leq d$.

Important Maclaurin series representations.

Function	Validity	Function	Validity
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$-1 < x < 1$	$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	all x
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	all x	$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	all x
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$-1 < x \leq 1$	$\arctan x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \leq x \leq 1$

¹The term "Maclaurin series" has a peculiar status: it essentially exists only in calculus courses. People who use power series regularly, in math or physics, speak instead about a Taylor series or power series at 0.

Example: (1.) Compute the Taylor series for $f(x) = \ln(x)$ at $a = 10$, and (2.) use Taylor's inequality to show when $|x - 10| \leq 4$ that $|R_n(x)| = |\ln(x) - T_n(x)| \rightarrow 0$ as $n \rightarrow \infty$.

Think: Differentiate $\ln x$ enough times to see a pattern. The pattern will give us the coefficients in the Taylor series and help us bound $|f^{(n+1)}(x)|$ to find M in Taylor's inequality.

Doing the problem: The first several higher derivatives of $f(x) = \ln x$ are in the table below.

n	0	1	2	3	4	5	6	7
$f^{(n)}(x)$	$\ln x$	$1/x$	$-1/x^2$	$2/x^3$	$-6/x^4$	$24/x^5$	$-120/x^6$	$720/x^7$

The pattern for $n \geq 1$ is $f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}$, so the Taylor series of $\ln x$ at $a = 10$ is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(10)}{n!} (x-10)^n &= f(10) + \sum_{n=1}^{\infty} \frac{f^{(n)}(10)}{n!} (x-10)^n \\ &= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (n-1)!}{10^n n!} (x-10)^n \\ &= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-10)^n}{10^n n} \\ &= \ln 10 + \frac{x-10}{10} - \frac{(x-10)^2}{200} + \frac{(x-10)^3}{3000} - \frac{(x-10)^4}{40000} + \dots \end{aligned}$$

Now find an M so that $|f^{(n+1)}(x)| \leq M$ when $|x - 10| \leq 4$, which means $6 \leq x \leq 14$.

Since $f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}$, we need $\left| \frac{n!}{x^{n+1}} \right| \leq M$ for $6 \leq x \leq 14$. The biggest value of

$$\left| \frac{n!}{x^{n+1}} \right| = \frac{n!}{x^{n+1}} \text{ in that } x\text{-range is } \frac{n!}{6^{n+1}}, \text{ so use } \boxed{M = \frac{n!}{6^{n+1}}}: \quad \text{if } |x - 10| \leq 4 \text{ then}$$

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-10|^{n+1} = \frac{n!/6^{n+1}}{(n+1)!} |x-10|^{n+1} = \frac{1}{n+1} \left(\frac{|x-10|}{6} \right)^{n+1} \leq \frac{(2/3)^{n+1}}{n+1}.$$

Thus $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$, so for $|x - 10| \leq 4$, $\ln x$ equals its Taylor series at $a = 10$.

Solutions should show all of your work, not just a single final answer.

1. Let $f(x) = \sqrt{x}$.

(a) Does $f(x)$ have a Maclaurin series? Why or why not?

(b) Determine the 3rd-degree Taylor polynomial $T_3(x)$ for $f(x) = \sqrt{x}$ at $a = 9$. Start off by filling in the following table of higher derivatives for $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0		
1		
2		
3		

(c) Compute $T_3(10)$ from (b). (This is an estimate for $\sqrt{10}$.)

(d) Use Taylor's inequality to bound the error $|\sqrt{10} - T_3(10)|$.

(e) Use a computing tool to confirm that the error is smaller than the error bound you stated in part (d).

2. Use the Maclaurin series for e^x and $\arctan x$ to find the Maclaurin series for the following functions. Determine the radius of convergence in each case.

(a) $f(x) = e^{3x} + e^{-3x}$

(b) $f(x) = \arctan\left(\frac{x}{3}\right)$

3. T/F (with justification)

If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \dots$ for $|x| < 1$ then $f'''(0) = 30$.