Recall (fill in the blank, see middle of pg 761)

The n-th Taylor Polynomial centered at a is

____.

Remainder in a Taylor Polynomial

Taylor polynomials provide good approximations to functions near a specific point, but how good are the approximations?

Let $R_n(x) = f(x) - T_n(x)$, then $R_n(x)$ is called the remainder of the Taylor series.

(Copy from pg 762) Theorem Taylor's Inequality

Suppose there exists a number M such that

$$\left|f^{(n+1)}(x)\right| \le M \text{ for } |x-a| \le d,$$

then the remainder $R_n(x)$ of the Taylor series satisfies

$|R_n(x)| \le$ ____________________ for ____________________.

Follow Sec 11.11 (next section) Example 1 pg 775-776: Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the **Taylor polynomials of order 2** centered at $x = 8$ for $f(x)$.

b. How accurate is this approximation when $7 \le x \le 9$?

Using Taylor series to solve other problems.

1. The Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $-\infty < x < \infty$.

   Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$ using Maclaurin series.

   Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$ using L'hospital rule.

   Verify your result $\lim_{x \to 0} \frac{e^x - 1}{x}$ using a graphing tool if you have one.

2. The Maclaurin series for $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for $-\infty < x < \infty$.

   Assume the conditions for the **Integral Test** have been verified. Use the Integral Test (although other methods are possible) to determine the convergence or divergence of the series $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$.

[Solution of Problem 2]

Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \geq 1$.

Note that $0 < \frac{1}{x} \leq 1 < \frac{\pi}{2}$, so $\frac{1}{x}$ is in the first quadrant.
Hence $f(x)$ is positive for $x \geq 1$.
In addition, $f(x)$ is continuous for $x \geq 1$.
Furthermore, $f'(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right) < 0$ for $x \geq 1$.
Thus $f(x)$ is decreasing for $x \geq 1$.

Therefore, the Integral Test applies.

For $x \geq 1$,
$$\sin\left(\frac{1}{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{x}\right)^{2n+1} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{-2n-1}$$

Thus
$$\int \sin\left(\frac{1}{x}\right) dx = \int \left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{-2n-1}\right] dx = \ln|x| - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!} x^{-2n} + C$$

Therefore,
$$\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx = \lim_{t\to\infty} \int_1^t \sin\left(\frac{1}{x}\right) dx$$
$$= \lim_{t\to\infty} \left[\ln|x| - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!} x^{-2n}\right]_1^t$$
$$= \lim_{t\to\infty} \left[\ln|t| - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!}\frac{1}{t^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!}\right]$$

Note that $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!}$ is absolutely convergent by the Ratio Test since

$$\lim_{x\to\infty} \frac{\frac{1}{(2n+2)(2n+3)!}}{\frac{1}{2n(2n+1)!}} = \lim_{x\to\infty} \frac{2n(2n+1)!}{(2n+2)(2n+3)!} = \lim_{x\to\infty} \frac{n}{(n+1)(2n+3)(2n+2)} = 0$$

As a result, $\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx = \lim_{t\to\infty} \left[\ln|t| - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!} \frac{1}{t^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!} \right]$

$= \infty$

The improper integral $\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx$ diverges.

Therefore, the infinite series $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ diverges by the Integral Test.