

Homework (ANSWER KEY)

1. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$.

[Solution]

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^5 x \, dx &= \int_0^{\frac{\pi}{2}} (\sin^4 x) \sin x \, dx \\ &= \int_0^{\frac{\pi}{2}} (\sin^2 x)^2 \sin x \, dx \\ &= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)^2 \sin x \, dx\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x \, dx$.

When $x = 0$, $u = 1$ and when $x = \frac{\pi}{2}$, $u = 0$.

$$\begin{aligned}\text{Thus } \int_0^{\frac{\pi}{2}} \sin^5 x \, dx &= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)^2 \sin x \, dx \\ &= - \int_1^0 (1 - u^2)^2 \, du \\ &= \int_0^1 (1 - 2u^2 + u^4) \, du \\ &= \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right)_0^1 \\ &= \frac{8}{15}\end{aligned}$$

2. Evaluate $\int \frac{\cos^5 x}{\sin x} \, dx$.

[Solution]

$$\begin{aligned}\int \frac{\cos^5 x}{\sin x} \, dx &= \int \frac{\cos^4 x}{\sin x} \cos x \, dx \\ &= \int \frac{(\cos^2 x)^2}{\sin x} \cos x \, dx \\ &= \int \frac{(1 - \sin^2 x)^2}{\sin x} \cos x \, dx\end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$.

Homework (ANSWER KEY)

$$\begin{aligned}
 \text{Thus } \int \frac{\cos^5 x}{\sin x} dx &= \int \frac{(1-\sin^2 x)^2}{\sin x} \cos x dx \\
 &= \int \frac{(1-u^2)^2}{u} du \\
 &= \int \frac{1-2u^2+u^4}{u} du \\
 &= \int \left(\frac{1}{u} - 2u + u^3 \right) du \\
 &= \ln|u| - u^2 + \frac{1}{4}u^4 + C \\
 &= \ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C
 \end{aligned}$$

3. Evaluate $\int_0^\pi \cos^4 2x dx$.

[Solution]

$$\begin{aligned}
 \int_0^\pi \cos^4 2x dx &= \int_0^\pi (\cos^2 2x)^2 dx \\
 &= \int_0^\pi \left(\frac{1+\cos 4x}{2} \right)^2 dx \\
 &= \frac{1}{4} \int_0^\pi (1+2\cos 4x+\cos^2 4x) dx \\
 &= \frac{1}{4} \int_0^\pi \left(1+2\cos 4x+\frac{1+\cos 8x}{2} \right) dx \\
 &= \frac{1}{8} \int_0^\pi (3+4\cos 4x+\cos 8x) dx \\
 &= \frac{1}{8} \left(3x + \sin 4x + \frac{1}{8} \sin 8x \right)_0^\pi \\
 &= \frac{3}{8} \pi
 \end{aligned}$$

Homework (ANSWER KEY)

4. Evaluate $\int \sin^3 x \cos^5 x \, dx$.

[Solution] 1

$$\begin{aligned}\int \sin^3 x \cos^5 x \, dx &= \int (\sin^3 x \cos^4 x) \cos x \, dx \\ &= \int [\sin^3 x (\cos^2 x)^2] \cos x \, dx \\ &= \int [\sin^3 x (1 - \sin^2 x)^2] \cos x \, dx\end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$.

$$\begin{aligned}\text{Thus } \int \sin^3 x \cos^5 x \, dx &= \int [\sin^3 x (1 - \sin^2 x)^2] \cos x \, dx \\ &= \int u^3 (1 - u^2)^2 \, du \\ &= \int u^3 (1 - 2u^2 + u^4) \, du \\ &= \int (u^3 - 2u^5 + u^7) \, du \\ &= \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C \\ &= \frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C\end{aligned}$$

[Solution] 2

$$\begin{aligned}\int \sin^3 x \cos^5 x \, dx &= \int (\sin^2 x \cos^5 x) \sin x \, dx \\ &= \int [(1 - \cos^2 x) \cos^5 x] \sin x \, dx\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x \, dx$.

$$\begin{aligned}\text{Thus } \int \sin^3 x \cos^5 x \, dx &= \int [(1 - \cos^2 x) \cos^5 x] \sin x \, dx \\ &= - \int (1 - u^2) u^5 \, du \\ &= - \int (u^5 - u^7) \, du \\ &= - \frac{1}{6}u^6 + \frac{1}{8}u^8 + C \\ &= - \frac{1}{6}\cos^6 x + \frac{1}{8}\cos^8 x + C\end{aligned}$$

Homework (ANSWER KEY)

5. Evaluate $\int \sin^2 x \cos^2 x \, dx$.

[Solution]

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1-\cos 2x}{2} \right) \left(\frac{1+\cos 2x}{2} \right) dx \\ &= \frac{1}{4} \int (1 - \cos^2 2x) \, dx \\ &= \frac{1}{4} \int \sin^2 2x \, dx \\ &= \frac{1}{4} \int \frac{1-\cos 4x}{2} \, dx \\ &= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) + C \\ &= \frac{1}{8}x - \frac{1}{32} \sin 4x + C\end{aligned}$$

6. Evaluate $\int \tan^3 x \sec^3 x \, dx$.

[Solution]

$$\begin{aligned}\int \tan^3 x \sec^3 x \, dx &= \int (\tan^2 x \sec^2 x) \tan x \sec x \, dx \\ &= \int [(\sec^2 x - 1) \sec^2 x] \tan x \sec x \, dx\end{aligned}$$

Let $u = \sec x$, then $du = \tan x \sec x \, dx$.

$$\begin{aligned}\text{Thus } \int \tan^3 x \sec^3 x \, dx &= \int [(\sec^2 x - 1) \sec^2 x] \tan x \sec x \, dx \\ &= \int (u^2 - 1) u^2 \, du \\ &= \int (u^4 - u^2) \, du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C\end{aligned}$$

7. Evaluate $\int \tan^2 x \sec^4 x \, dx$.

[Solution]

$$\int \tan^2 x \sec^4 x \, dx = \int (\tan^2 x \sec^2 x) \sec^2 x \, dx$$

Homework (ANSWER KEY)

$$= \int [\tan^2 x (\tan^2 x + 1)] \sec^2 x \, dx$$

Let $u = \tan x$, then $du = \sec^2 x \, dx$.

$$\begin{aligned} \text{Thus } \int \tan^2 x \sec^4 x \, dx &= \int [\tan^2 x (\tan^2 x + 1)] \sec^2 x \, dx \\ &= \int u^2 (u^2 + 1) \, du \\ &= \int (u^4 + u^2) \, du \\ &= \frac{1}{5} u^5 + \frac{1}{3} u^3 + C \\ &= \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C \end{aligned}$$

8. Evaluate $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$.

[Solution]

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sec^4 x \, dx &= \int_0^{\frac{\pi}{4}} (\sec^2 x) \sec^2 x \, dx \\ &= \int_0^{\frac{\pi}{4}} (\tan^2 x + 1) \sec^2 x \, dx \end{aligned}$$

Let $u = \tan x$, then $du = \sec^2 x \, dx$.

When $x = 0$, $u = 0$ and when $x = \frac{\pi}{4}$, $u = 1$.

$$\begin{aligned} \text{Therefore, } \int_0^{\frac{\pi}{4}} \sec^4 x \, dx &= \int_0^{\frac{\pi}{4}} (\tan^2 x + 1) \sec^2 x \, dx \\ &= \int_0^1 (u^2 + 1) \, du \\ &= \left(\frac{1}{3} u^3 + u \right)_0^1 \\ &= \frac{4}{3} \end{aligned}$$

Homework (ANSWER KEY)

9. Evaluate $\int \frac{1-\tan^2 x}{1+\tan^2 x} dx$.

[Solution]

$$\int \frac{1-\tan^2 x}{1+\tan^2 x} dx \text{ (apply the identity } \tan^2 x + 1 = \sec^2 x \text{ to the denominator)}$$

(rewrite $\sec^2 x$ as $\cos^2 x$)(apply the identity $\cos^2(2x) = \cos^2 x - \sin^2 x$)

$$= \int \cos 2x dx$$

$$= \frac{1}{2} \sin 2x + C$$

10. Evaluate $\int \tan^3 x \sec^4 x dx$.

[Solution] 1

$$\begin{aligned} \int \tan^3 x \sec^4 x dx &= \int (\tan^3 x \sec^2 x) \sec^2 x dx \\ &= \int [\tan^3 x (\tan^2 x + 1)] \sec^2 x dx \end{aligned}$$

Let $u = \tan x$, then $du = \sec^2 x dx$.

$$\begin{aligned} \text{Thus } \int \tan^3 x \sec^4 x dx &= \int [\tan^3 x (\tan^2 x + 1)] \sec^2 x dx \\ &= \int u^3 (u^2 + 1) du \\ &= \int (u^5 + u^3) du \\ &= \frac{1}{6} u^6 + \frac{1}{4} u^4 + C \\ &= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C \end{aligned}$$

[Solution] 2

$$\begin{aligned} \int \tan^3 x \sec^4 x dx &= \int (\tan^2 x \sec^3 x) \tan x \sec x dx \\ &= \int [(\sec^2 x - 1) \sec^3 x] \tan x \sec x dx \end{aligned}$$

Let $u = \sec x$, then $du = \tan x \sec x dx$.

$$\begin{aligned} \text{Thus } \int \tan^3 x \sec^4 x dx &= \int [(\sec^2 x - 1) \sec^3 x] \tan x \sec x dx \\ &= \int (u^2 - 1) u^3 du \\ &= \int (u^5 - u^3) du \\ &= \frac{1}{6} u^6 - \frac{1}{4} u^4 + C \\ &= \frac{1}{6} \sec^6 x - \frac{1}{4} \sec^4 x + C \end{aligned}$$