

HW (ANSWER KEY)

1. Evaluate $\int \frac{1}{x\sqrt{4-x^2}} dx.$

[Solution]

Let $x = 2 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $dx = 2 \cos \theta d\theta$.

$$\begin{aligned} \text{Thus } \int \frac{1}{x\sqrt{4-x^2}} dx &= \int \frac{1}{2\sin\theta\sqrt{4-4\sin^2\theta}} 2\cos\theta d\theta \\ &= \frac{1}{2} \int \frac{1}{\sin\theta} d\theta \\ &= \frac{1}{2} \int \csc\theta d\theta \\ &= -\frac{1}{2} \ln|\csc\theta + \cot\theta| + C \\ &= -\frac{1}{2} \ln \left| \frac{2 + \sqrt{4-x^2}}{x} \right| + C \end{aligned}$$

2. Evaluate $\int \frac{1}{\sqrt{x^2+16}} dx.$

[Solution]

Let $x = 4 \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = 4 \sec^2 \theta d\theta$.

$$\begin{aligned} \text{Thus } \int \frac{1}{\sqrt{x^2+16}} dx &= \int \frac{1}{\sqrt{16\tan^2\theta+16}} 4\sec^2\theta d\theta \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln \left| \frac{\sqrt{x^2+16}+x}{4} \right| + C \end{aligned}$$

3. Evaluate $\int_{\sqrt{2}}^2 \frac{1}{x^3\sqrt{x^2-1}} dx.$

[Solution]

Let $x = \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$, then $dx = \sec \theta \tan \theta d\theta$.

When $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$ and when $x = 2$, $\theta = \frac{\pi}{3}$.

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$$\begin{aligned}
 \text{Thus } \int_{\sqrt{2}}^2 \frac{1}{x^3 \sqrt{x^2 - 1}} dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^3 \theta \sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2 \theta} d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 \theta d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1 + \cos 2\theta}{2} d\theta \\
 &= \left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
 &= \left(\frac{\pi}{6} + \frac{1}{4}\sin \frac{2\pi}{3} \right) - \left(\frac{\pi}{8} + \frac{1}{4}\sin \frac{\pi}{2} \right) \\
 &= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}
 \end{aligned}$$

4. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx$.

[Solution]

Let $u = \sin x$, then $du = \cos x dx$.

When $x = 0$, $u = 0$ and when $x = \frac{\pi}{2}$, $u = 1$.

$$\text{Thus } \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx = \int_0^1 \frac{1}{\sqrt{1 + u^2}} du$$

Let $u = \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $du = \sec^2 \theta d\theta$.

When $u = 0$, $\theta = 0$ and when $u = 1$, $\theta = \frac{\pi}{4}$.

$$\begin{aligned}
 \text{Thus } \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx &= \int_0^1 \frac{1}{\sqrt{1 + u^2}} du \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1 + \tan^2 \theta}} \sec^2 \theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} \sec \theta d\theta \\
 &= (\ln |\sec \theta + \tan \theta|) \Big|_0^{\frac{\pi}{4}} \\
 &= \ln(\sqrt{2} + 1)
 \end{aligned}$$

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5. Let R be the region bounded by the function $f(x) = \frac{1}{\sqrt{-x^2 + 2x + 3}}$ and x -axis on the interval $[0, 1]$.

Find the area of the region R .

[Solution] 1

$$\begin{aligned} A &= \int_0^1 \frac{1}{\sqrt{-x^2 + 2x + 3}} dx \\ -x^2 + 2x + 3 &= -(x^2 - 2x) + 3 \\ &= -(x^2 - 2x + 1) + 1 + 3 \\ &= -(x-1)^2 + 4 \end{aligned}$$

$$\begin{aligned} \text{Thus } A &= \int_0^1 \frac{1}{\sqrt{-x^2 + 2x + 3}} dx \\ &= \int_0^1 \frac{1}{\sqrt{4 - (x-1)^2}} dx \end{aligned}$$

Let $x-1 = 2 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $dx = 2 \cos \theta d\theta$.

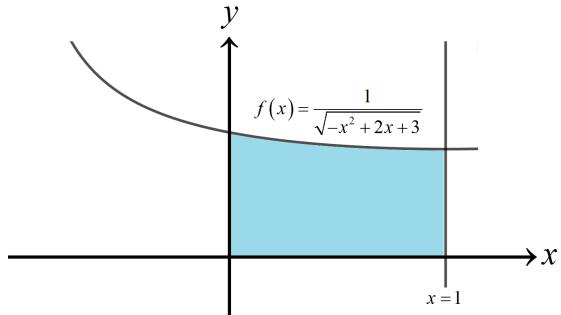
When $x=0$, $\theta = -\frac{\pi}{6}$ and when $x=1$, $\theta = 0$.

$$\begin{aligned} \text{Thus } A &= \int_0^1 \frac{1}{\sqrt{4 - (x-1)^2}} dx \\ &= \int_{-\frac{\pi}{6}}^0 \frac{1}{\sqrt{4 - 4 \sin^2 \theta}} 2 \cos \theta d\theta \\ &= \int_{-\frac{\pi}{6}}^0 1 d\theta \\ &= \frac{\pi}{6} \end{aligned}$$

[Solution] 2

$$\begin{aligned} A &= \int_0^1 \frac{1}{\sqrt{4 - (x-1)^2}} dx \\ &= \int_0^1 \frac{1}{2\sqrt{1 - \left(\frac{x-1}{2}\right)^2}} dx \end{aligned}$$

Let $u = \frac{x-1}{2}$, then $du = \frac{1}{2} dx$.



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When $x = 0$, $u = -\frac{1}{2}$ and when $x = 1$, $u = 0$.

$$\begin{aligned} \text{Thus } A &= \int_0^1 \frac{1}{2\sqrt{1-\left(\frac{x-1}{2}\right)^2}} dx \\ &= \int_{-\frac{1}{2}}^0 \frac{1}{\sqrt{1-u^2}} du \\ &= \left(\sin^{-1} u\right)_{-\frac{1}{2}}^0 \\ &= \frac{\pi}{6} \end{aligned}$$

6. Evaluate $\int \frac{1}{\sqrt{1+16x^2}} dx$.

[Solution]

Let $x = \frac{1}{4} \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = \frac{1}{4} \sec^2 \theta d\theta$.

$$\begin{aligned} \text{Thus } \int \frac{1}{\sqrt{1+16x^2}} dx &= \int \frac{1}{\sqrt{1+\tan^2 \theta}} \frac{1}{4} \sec^2 \theta d\theta \\ &= \frac{1}{4} \int \sec \theta d\theta \\ &= \frac{1}{4} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{4} \ln \left| \sqrt{1+16x^2} + 4x \right| + C \end{aligned}$$

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7. Evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} dx$.

[Solution]

Let $x = 2 \sin \theta$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $dx = 2 \cos \theta d\theta$.

When $x = 0$, $\theta = 0$ and when $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$.

$$\begin{aligned} \text{Thus } \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} dx &= \int_0^{\frac{\pi}{4}} \frac{4 \sin^2 \theta}{\sqrt{4-4 \sin^2 \theta}} 2 \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} 4 \sin^2 \theta d\theta \\ &= 4 \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2} d\theta \\ &= (2\theta - \sin 2\theta) \Big|_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

8. Evaluate $\int \frac{1}{x^2 \sqrt{9x^2-1}} dx$.

[Solution]

Let $x = \frac{1}{3} \sec \theta$ where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$, then $dx = \frac{1}{3} \sec \theta \tan \theta d\theta$.

$$\begin{aligned} \text{Thus } \int \frac{1}{x^2 \sqrt{9x^2-1}} dx &= \int \frac{1}{\frac{1}{9} \sec^2 \theta \sqrt{\sec^2 \theta - 1}} \frac{1}{3} \sec \theta \tan \theta d\theta \\ &= 3 \int \frac{1}{\sec \theta} d\theta \\ &= 3 \int \cos \theta d\theta \\ &= 3 \sin \theta + C \\ &= \frac{\sqrt{9x^2-1}}{x} + C \end{aligned}$$

HW (ANSWER KEY)

9. Evaluate $\int \sqrt{5+4x-x^2} dx$.

[Solution]

$$\begin{aligned} 5+4x-x^2 &= -(x^2 - 4x) + 5 \\ &= -(x^2 - 4x + 4) + 4 + 5 \\ &= -(x-2)^2 + 9 \end{aligned}$$

Let $x-2 = 3\sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $dx = 3\cos\theta d\theta$.

$$\begin{aligned} \text{Thus } \int \sqrt{5+4x-x^2} dx &= \int \sqrt{9-(x-2)^2} dx \\ &= \int 3\cos\theta \sqrt{9-9\sin^2\theta} d\theta \\ &= 9 \int \cos^2\theta d\theta \\ &= 9 \int \frac{1+\cos 2\theta}{2} d\theta \\ &= \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + C \\ &= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2} \cdot \frac{x-2}{3} \cdot \frac{\sqrt{9-(x-2)^2}}{3} + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{x-2}{2}\sqrt{9-(x-2)^2} + C \end{aligned}$$