ANSWER KEY for homework

Name : ______________________________

1. Evaluate $\int \ln\left(x+\sqrt{1+x^2}\right)\,dx$.

[Solution]

Let $u=\ln\left(x+\sqrt{1+x^2}\right)$ and $dv=dx$,

$$\text{then } du=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)dx$$
$$=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\,dx$$
$$=\frac{1}{\sqrt{1+x^2}}\,dx$$

and $v=x$.

Thus $\int \ln\left(x+\sqrt{1+x^2}\right)dx = x\ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x}{\sqrt{1+x^2}}\,dx$

For $\int\frac{x}{\sqrt{1+x^2}}\,dx$, let $w=1+x^2$, then $dw=2x\,dx$.

$$\text{Thus } \int\frac{x}{\sqrt{1+x^2}}\,dx=\int\frac{1}{2\sqrt{w}}\,dw$$
$$=\sqrt{w}+C$$
$$=\sqrt{1+x^2}+C$$

Therefore, $\int \ln\left(x+\sqrt{1+x^2}\right)dx = x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C$

2. Evaluate $\int x\tan^2 x\,dx$.

[Solution]

$$\int x\tan^2 x\,dx=\int x\left(\sec^2 x-1\right)dx$$
$$=\int x\sec^2 x\,dx-\int x\,dx$$

For $\int x\sec^2 x\,dx$, let $u=x$ and $dv=\sec^2 x\,dx$,

then $du=dx$ and $v=\tan x$.

$$\text{Thus } \int x\sec^2 x\,dx=x\tan x-\int\tan x\,dx$$
$$=x\tan x-\int\frac{\sin x}{\cos x}\,dx$$

For $\int \frac{\sin x}{\cos x}\,dx$, let $w = \cos x$, then $dw = -\sin x\,dx$.

Thus $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$$= -\int \frac{1}{w}\,dw$$

$$= -\ln|w| + C$$

$$= -\ln|\cos x| + C$$

Therefore, $\int x\tan^2 x\,dx = \int x\sec^2 x\,dx - \int x\,dx$

$$= \left(x\tan x - \int \tan x\,dx\right) - \frac{1}{2}x^2$$

$$= x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

3. Evaluate $\int \cos\sqrt{x}\,dx$

[Solution]

Let $w = \sqrt{x}$, then $dw = \frac{1}{2\sqrt{x}}\,dx$.

Thus $\int \cos\sqrt{x}\,dx = 2\int w\cos w\,dw$

For $\int w\cos w\,dw$, let $u = w$ and $dv = \cos w\,dw$,

then $du = dw$ and $v = \sin w$.

Thus $\int w\cos w\,dw = w\sin w - \int \sin w\,dw$

$$= w\sin w + \cos w + c$$

Therefore, $\int \cos\sqrt{x}\,dx =$ 2 [sqrt(x) sin (sqrt(x) + cos (sqrt(x))] + C

4. Evaluate $\int x^2 (\ln x)^2 \, dx$.

[Solution]

Let $u = (\ln x)^2$ and $dv = x^2 \, dx$,

then $du = \dfrac{2\ln x}{x} \, dx$ and $v = \dfrac{1}{3}x^3$.

Thus $\int x^2 (\ln x)^2 \, dx = \dfrac{1}{3}x^3 (\ln x)^2 - \dfrac{2}{3}\int x^2 \ln x \, dx$

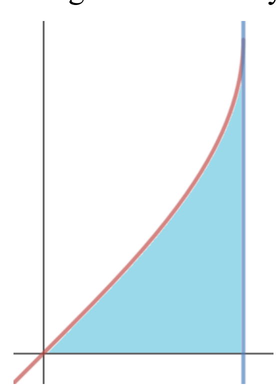
For $\int x^2 \ln x \, dx$, let $s = \ln x$ and $dt = x^2 \, dx$,

then $ds = \dfrac{1}{x} \, dx$ and $t = \dfrac{1}{3}x^3$.

Thus $\int x^2 \ln x \, dx = \dfrac{1}{3}x^3 \ln x - \dfrac{1}{3}\int x^2 \, dx$

Therefore,
$$\begin{aligned}\int x^2 (\ln x)^2 \, dx &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\left(\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx\right) \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C\end{aligned}$$

5. Consider the graph of the function $f(x) = \sin^{-1} x$. Let R be the region bounded by $y = f(x)$ and x-axis on the interval $[0,1]$.



Evaluate the **area** of R.

[Solution]

$A = \int_0^1 \sin^{-1} x \, dx$

Let $u = \sin^{-1} x$ and $dv = dx$,

then $du = \dfrac{1}{\sqrt{1-x^2}} \, dx$ and $v = x$.

Thus $\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \dfrac{x}{\sqrt{1-x^2}} \, dx$

For $\int \frac{x}{\sqrt{1-x^2}}\,dx$, let $w=1-x^2$, then $dw=-2x\,dx$.

Thus $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{2\sqrt{w}}\,dw$

$$= -\sqrt{w}+C$$
$$= -\sqrt{1-x^2}+c$$

Therefore, $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$$= x\sin^{-1}x+\sqrt{1-x^2}+C$$

As a result, $A = \int_0^1 \sin^{-1}x\,dx$

$$= \left(x\sin^{-1}x+\sqrt{1-x^2}\right)\Big|_0^1$$
$$= \sin^{-1}1 - 1$$
$$= \frac{\pi}{2}-1$$

6. Evaluate $\int \cos(\ln x)\,dx$.

[Solution]

Let $u=\cos(\ln x)$ and $dv=dx$,

then $du=\frac{-\sin(\ln x)}{x}\,dx$ and $v=x$.

Thus $\int \cos(\ln x)\,dx = x\cos(\ln x)+\int \sin(\ln x)\,dx$

For $\int \sin(\ln x)\,dx$, let $s=\sin(\ln x)$ and $dt=dx$,

then $ds=\frac{\cos(\ln x)}{x}\,dx$ and $t=x$.

Thus $\int \sin(\ln x)\,dx = x\sin(\ln x)-\int \cos(\ln x)\,dx$

Therefore, $\int \cos(\ln x)\,dx = x\cos(\ln x)+\int \sin(\ln x)\,dx$

$$= x\cos(\ln x)+\left[x\sin(\ln x)-\int \cos(\ln x)\,dx\right]$$

As a result, $\int \cos(\ln x)\,dx = \frac{x\cos(\ln x)+x\sin(\ln x)}{2}+C$

7. There are parts (a)-(f).

a) To derive the formula for **Integration by Parts** we used which of the following theorems?

1) The Fundamental Theorem of Calculus.

2) The Product Rule of Differentiation.

3) The Chain Rule of Differentiation.

4) The Mean Value Theorem

b) Evaluate $\int_0^{\frac{\pi}{2}} x\cos 2x\, dx$.

[Solution]

Let $u = x$ and $dv = \cos 2x\, dx$,

then $du = dx$ and $v = \frac{1}{2}\sin 2x$.

Thus $\int_0^{\frac{\pi}{2}} x\cos 2x\, dx = \frac{1}{2}(x\sin 2x)_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

$$= \frac{1}{2}\left[\frac{\pi}{2}\sin\pi - 0\right] - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}(\cos\pi - \cos 0)$$

$$= -\frac{1}{2}$$

c) Suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$ and f'' is continuous. Evaluate $\int_1^4 x f''(x)\,dx$.

[Solution]

Let $u = x$ and $dv = f''(x)\,dx$,

then $du = dx$ and $v = f'(x)$.

Thus
$$\int_1^4 x f''(x)\,dx = xf'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx$$
$$= \left[4f'(4) - f'(1)\right] - \left[f(4) - f(1)\right]$$
$$= 2$$

d) Evaluate $\int \tan^{-1} x\,dx$.

[Solution]

Let $u = \tan^{-1} x$ and $dv = dx$,

then $du = \frac{1}{1+x^2}\,dx$ and $v = x$.

Thus $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$.

For $\int \frac{x}{1+x^2}\,dx$, let $w = 1+x^2$, then $dw = 2x\,dx$.

Therefore
$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int \frac{1}{w}\,dw$$
$$= x\tan^{-1} x - \frac{1}{2}\ln|w| + C$$
$$= x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C$$

Note that $\ln\left|1+x^2\right| = \ln\left(1+x^2\right)$ here because $1+x^2 > 0$ for all real numbers x.

e) Evaluate $\int e^x \cos x \, dx$.

[Solution]

Let $u = e^x$ and $dv = \cos x \, dx$

Then $du = e^x \, dx$ and $v = \sin x$

Thus $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

For $\int e^x \sin x \, dx$

Let $s = e^x$ and $dt = \sin x \, dx$

Then $ds = e^x \, dx$ and $t = -\cos x$

Thus $\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

As a result,
$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right] \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx\end{aligned}$$

Therefore, $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C$

[Alternative Solution]

Let $u = \cos x$ and $dv = e^x \, dx$

Then $du = -\sin x \, dx$ and $v = e^x$

Thus $\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$

For $\int e^x \sin x \, dx$

Let $s = \sin x$ and $dt = e^x \, dx$

Then $ds = \cos x \, dx$ and $t = e^x$

Thus $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

As a result,
$$\begin{aligned}\int e^x \cos x \, dx &= e^x \cos x + \int e^x \sin x \, dx \\ &= e^x \cos x + \left[e^x \sin x - \int e^x \cos x \, dx \right] \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx\end{aligned}$$

Therefore, $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C$

f) A particle that moves along a straight line has velocity $v(t)=t^3e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

[Solution]

The particle will move $s(t)=\int_0^t v(x)\,dx$ meters for the first t seconds.

For $\int x^3e^{-x}\,dx$

Let $u=x^3$ and $dv=e^{-x}\,dx$

Then $du=3x^2\,dx$ and $v=-e^{-x}$

Thus $\int x^3e^{-x}\,dx=-x^3e^{-x}+3\int x^2e^{-x}\,dx$

For $\int x^2e^{-x}\,dx$

Let $m=x^2$ and $dn=e^{-x}\,dx$

Then $dm=2x\,dx$ and $n=-e^{-x}$

Thus $\int x^2e^{-x}\,dx=-x^2e^{-x}+2\int xe^{-x}\,dx$

Therefore,
$$\begin{aligned}\int x^3e^{-x}\,dx&=-x^3e^{-x}+3\int x^2e^{-x}\,dx\\&=-x^3e^{-x}+3\left[-x^2e^{-x}+2\int xe^{-x}\,dx\right]\\&=-x^3e^{-x}-3x^2e^{-x}+6\int xe^{-x}\,dx\end{aligned}$$

For $\int xe^{-x}\,dx$

Let $f=x$ and $dg=e^{-x}\,dx$

Then $df=dx$ and $g=-e^{-x}$

Thus $\int xe^{-x}\,dx=-xe^{-x}+\int e^{-x}\,dx$

Therefore,
$$\begin{aligned}\int x^3e^{-x}\,dx&=-x^3e^{-x}-3x^2e^{-x}+6\int xe^{-x}\,dx\\&=-x^3e^{-x}-3x^2e^{-x}+6\left[-xe^{-x}+\int e^{-x}\,dx\right]\\&=-x^3e^{-x}-3x^2e^{-x}-6xe^{-x}+6\int e^{-x}\,dx\\&=-x^3e^{-x}-3x^2e^{-x}-6xe^{-x}-6e^{-x}+C\end{aligned}$$

As a result,
$$\begin{aligned}s(t)&=\int_0^t v(x)\,dx\\&=\left(-t^3e^{-t}-3t^2e^{-t}-6te^{-t}-6e^{-t}+C\right)-\left(-6+C\right)\\&=-t^3e^{-t}-3t^2e^{-t}-6te^{-t}-6e^{-t}+6\end{aligned}$$