(Answer KEY)

Name : ________________________________

NOTE: FIRST ATTEMPT WITHOUT LOOKING AT THE HINTS AND ANSWERS.
Check your answer on Wolfram|Alpha by typing "series representation for ..."

1. Suppose the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is R. Use the **Test of Your Choice** to find the radius of convergence of the series.

   a. $\sum_{n=1}^{\infty} n c_n x^{n-1}$

   b. $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$

   ANSWER: See Theorem 2 Sec 11.9 pg 754.

2. Find a power series representation for the function and determine the interval of convergence.

   a. $\frac{5}{1-4x^2}$

   b. $\frac{2}{3-x}$

   c. $\frac{x}{9+x^2}$

   d. $\frac{x}{2x^2+1}$

   GUIDE FOR a,b,c,d: Follow Examples 1,2,3 Sec 11.9.

   e. $\frac{3}{x^2-x-2}$

   f. $\frac{x+2}{2x^2-x-1}$

   GUIDE FOR e, f: First use partial fraction. Compute the two power series using the same method as Example 2. Then combine the two series.

   g. $\frac{x}{(1+4x)^2}$

   ANSWER:
   Step i: Apply Theorem 2 (term-by-term differentiation) to get the series for -4/(1+4x)^2.
   Step ii: Then multiply the series by x/(-4).
   Step iii: Radius of convergence is R = 1/4.

h. $\left(\frac{x}{2-x}\right)^3$

ANSWER:
Step i: To get 1/(2-x)^2, follow Example 5.
Step ii: Apply another term-by-term differentiation to get 1/(2-x)^3.
Step iii: Multiply this series by (x^3)/2.
Step iv: Radius of converge is R = 2.

i. $\frac{1+x}{(1-x)^2}$

ANSWER:
Step i: To get 1/(1-x)^2, use Example 5.
Step ii: To get x/(1-x)^2, multiply the series by x.
Step iii: Combine them, then shift some indices to turn them into just one series.
Step iv: Radius of convergence is R = 1.

3. Evaluate the indefinite integral as a power series. What is the radius of convergence?

a. $\int \frac{x}{1+x^3}\,dx$

ANSWER:
Step i: First follow Example 8a (top half) to get the series for 1/(1+x^3).
Step ii: Then multiply your series by x.
Step iii: Then apply Theorem 2 (term-by-term integration).
Step iv: Radius of convergence is the same as in Example 8a, R = 1.

b. $\int x^2 \ln(1+x)\,dx$

ANSWER:
Step i: First follow Example 6 to get the series for ln(1+x).
Step ii: Then multiply your series by x^2.
Step iii: Then apply Theorem 2 (term-by-term integration).
Step iv: Radius of convergence is the same as in Example 6, R = 1.

c. $\int \frac{\tan^{-1} x}{x}\,dx$

ANSWER:
Step i: First follow Example 7 to get the series for arctan(x).
Step ii: Then multiply your series by 1/x.
Step iii: Then apply Theorem 2 (term-by-term integration).
Step iv: Radius of convergence is the same as in Example 7, R = 1.

4. Consider the geometric series $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$ for $|x|<1$.

   a. Find the sum of the series $\sum_{n=1}^{\infty} nx^n$ for $|x|<1$.

   ANSWER:
   Step i: Use Theorem 2 (term-by-term differentiation) to get 1/(1-x)^2
   Step ii: Then multiply by x.
   Step iii: Answer is x/(1-x)^2

   b. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

   ANSWER:
   Step i: Put x = 1/2 into the function for part (a).
   Step ii: You get (1/2)/(1 - 1/2)^2 = 2.

   c. Find the sum of the series $\sum_{n=1}^{\infty} n^2x^n$ for $|x|<1$.

   ANSWER: Start with the answer for part (a). Use Theorem 2 (term-by-term differentiation), then multiply by x.

   d. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

   ANSWER: Put x = 1/2 into the function for part (d).

5. It is known that $\cos x$ has a power series representation $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

   a. Find a power series representation for $\cos\sqrt{x}$.
   ANSWER: Use Theorem (composition) with h(x) = sqrt(x) and f(x) = cos x. The answer should look like the same series but you replace x^(2n) with x^n.

   b. Find a power series representation for $\int \cos\sqrt{x}\, dx$.
   GUIDE: Apply Theorem 2 (term-by-term integration).

   c. Assume that the series you found in part (b) converges for all $x \geq 0$. Use your answer in part (b) to determine a series that represents $\int_0^1 \cos\sqrt{x}\, dx$.
   GUIDE: Follow Example 8b(top half). The answer should be a series, not an approximation.

   d. If the first two non-zero terms of the series are used to estimate the value of the definite integral from part (c), provide a bound on the error of this estimate.
   GUIDE: Use Alternating Series Theorem. Follow Example 8b(bottom half).

6. It is known that e^x has a power series representation $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $-\infty < x < \infty$.

   a. Find a power series representation for xe^x.
   ANSWER: Multiply the given series by the number x. Answer is sum from n=0 of x^(1+n)/n!

   b. Find a power series representation for $\frac{d}{dx}\left(xe^x\right)$.
   ANSWER: Take the power series from part (a) and apply Theorem 2 (term-by-term differentiation). Answer is sum from n=0 of (n+1) x^n/n!

   c. Evaluate $\sum_{n=0}^{\infty} \frac{(n+1)(-1)^n}{n!}$.
   ANSWER:
   Step i: Evaluate the function for derivative d/dx (x e^x) using product rule - you should get (x+1)e^x.
   Step ii: Due to the power series of part (b), we know that this derivative when x = -1 is the sum of the given series (part c).
   Step iii: Final answer is ((-1) + 1) e^(-1) = 0.

   d. Find a power series representation for $\int xe^x \, dx$.
   ANSWER: Use the series from part (a) and apply Theorem 2 (term-by-term integration).
   Answer is sum from n = 0 of x^(n+2)/(n! (n+2)).

   e. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n!(n+2)}$.
   ANSWER:
   Step i: Evaluate the function for the antiderivative for the function x e^x using integration by parts - you should get (x-1) e^x + C.
   Step ii: Due to part (d), we know that this antiderivative when x = 1 is the sum of the series (part e) but starting at n=0, so the sum of the series (part e) starting at n=0 is equal the definite integral of x e^x from 0 to 1, which is equal to 1.
   Step iii: But the given series starts at n = 1, so we have to subtract the term where n = 0. The term when n= 0 is 1/(0! (0 + 2)) = 1/ (1 . 2) = 1/2.
   So the final answer is 1-1/2= 1/2.