11.3 The Integral Test and Estimates of Sums

The Integral Test. If $a_n = f(n)$ where $f(x)$ is a continuous, positive, decreasing function for $x \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. Another way of saying this is

(i) if $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges,

(ii) if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Tips on checking a function is decreasing: $f(x)$ is decreasing on an interval when $f'(x) < 0$ there. Algebra may also be useful: a sum or product of decreasing positive functions is decreasing, and if $f = 1/g(x)$ on an interval then $f(x)$ is decreasing there if $g(x)$ is increasing, which can be checked by showing $g'(x) > 0$ if it is not clear by algebra that $g(x)$ is increasing. For example, $\frac{1}{(x+1)(x+2)}$ is decreasing for $x \geq 1$ since $x+1$ and $x+2$ are each increasing for $x \geq 1$.

Remainder Estimate. If $a_n = f(n)$ for $f(x)$ as above, suppose $s = \sum_{n=1}^{\infty} a_n$ converges. For $N \geq 1$, let $s_N = \sum_{n=1}^{N} a_n$ and $R_N = s - s_N$, so R_N is the Nth remainder term. Then

$$\boxed{\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_N^{\infty} f(x)\,dx.}$$

Note: The integral test works for series not starting at $n = 1$: if $f(x)$ is continuous, positive, and decreasing for $x \geq c$ then convergence of $\sum_{n=c}^{\infty} f(n)$ is the same as convergence of $\int_c^{\infty} f(x)\,dx$.

Example: Determine whether or not $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges.

Thinking about the problem:

Use the integral test: $a_n = f(n)$ where $f(x) = \frac{x}{x^2+1}$. If we can show $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges exactly when $\int_1^{\infty} f(x)\,dx$ converges.

Doing the problem:

The function $f(x) = \dfrac{x}{x^2+1}$ is continuous since x and x^2+1 are continuous and the denominator is never 0 (no vertical asymptotes). For $x \geq 1$, x and x^2+1 are positive so $f(x) > 0$. To show $f(x)$ is decreasing for $x \geq 1$, its derivative (by the quotient rule) is $\dfrac{1-x^2}{(x^2+1)^2}$, and for $x > 1$ we have $1 - x^2 < 0$, so $f'(x) < 0$.[1] This confirms all the conditions needs to apply the integral test: to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges we look at $\displaystyle\int_1^{\infty} \frac{x}{x^2+1}\,dx$. Let $u = x^2+1$ so $du = 2x\,dx$, $x = 1 \Rightarrow u = 2$, and $x \to \infty \Rightarrow u \to \infty$, so

$$\begin{aligned}
\int_1^{\infty} \frac{x}{x^2+1}\,dx &= \int_2^{\infty} \frac{(1/2)du}{u} \\
&= \frac{1}{2}\int_2^{\infty} \frac{du}{u} \\
&= \frac{1}{2}\lim_{b\to\infty}\int_2^{b} \frac{du}{u} \\
&= \frac{1}{2}\lim_{b\to\infty}\left(\ln u\,\Big|_2^b\right) \\
&= \frac{1}{2}\lim_{b\to\infty}(\ln b - \ln 2) \\
&= \infty.
\end{aligned}$$

Since $\displaystyle\int_1^{\infty} \frac{x}{x^2+1}\,dx$ diverges, $\boxed{\text{the series } \displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} \text{ diverges}}$.

[1] The integral test for $\displaystyle\sum_{n=1}^{\infty} f(n)$ is valid provided $f(x)$ is *eventually* decreasing, not necessarily decreasing for $x \geq 1$, so it's not important to verify $f(x)$ decreases on $[1,\infty)$ rather than $(1,\infty)$, although it does.

Solutions should show all of your work, not just a single final answer.

1. Use the integral test to determine if $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges or diverges.

   (a) Explain why the integral test can be applied.

   (b) Apply the integral test to determine if $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges or diverges.

2. Use the integral test to determine if $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ converges or diverges. See the note on the first page of this worksheet about series not starting at $n = 1$.

   (a) Explain why the integral test can be applied.

   (b) Apply the integral test to determine if $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ converges or diverges.

3. Consider the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

   (a) Verify that the integral test can be used to decide if this series converges and then use it.

(b) Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

(c) Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N.

4. T/F (with justification): If $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n = \int_1^\infty f(x)\,dx$.