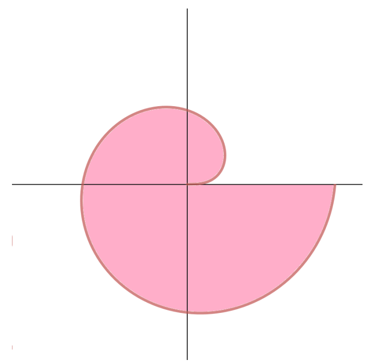
Homework Sec 10.4 (KEY)

Name : ______________________________

(Please use your own paper. Show all work. Leave plenty of space between each answer).

1. Find the area of the region that is bounded by the polar curve $r = \tan\theta$ and lies on the interval $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$. Hint: You can sketch the curve, but you don't need to know what the curve looks like - only that the function tan x is positive and nonnegative on the interval. Answer: sqrt(3)/3 - pi/12.

2. Find the area of the shaded region enclosed by the polar curve $r = \sqrt{\theta}$. Hint: You need to figure out first time the curve hits the positive polar axis. Answer: pi^2.



3. Find the area of the region enclosed by one loop of the polar curve $r = \cos 3\theta$. Hint: Similar to Example 1 pg 670. Answer: pi/12.

4. Find the area of the region enclosed by one loop of the polar curve $r = \sin 4\theta$. Answer: pi/16.

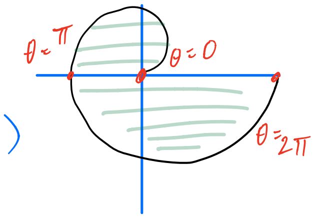
5. Find the area of the region inside the larger loop and outside the smaller loop of the polar curve $r = 1 + 2\cos\theta$. Hint: First, sketch the curve and mark the orientation. The orientation is counterclockwise on both the large loop and the small loop. The curve is symmetric about the polar axis, so it's enough to consider the upper half of the curve. Note that: when theta is 0, r is 3; when theta is 2pi/3, r is 0; when theta is pi, r is -1. So the integral for the larger loop should be taken from 0 and 2pi/3. The integral for the smaller loop should be taken from 2pi/3 to pi. There are other alternative bounds that would work as well! Answer: Area of larger loop is pi + 3/4 * sqrt(3). Area of smaller loop is pi/2 - 3/4*sqrt(3). Area of the total area (just above the polar axis) is [pi + ¾ * sqrt(3)] – [pi/2-3/4*sqrt(3)] = pi/2+3/2*sqrt(3). The total area of the region is twice the total area above the polar axis, which is 2*(pi/2+3/2*sqrt(3)) = pi + 3 * sqrt(3).

6. Find the area of the region that lies inside both $r = 4\sin 2\theta$ and $r = 4\cos 2\theta$. Answer: After graphing, you see there are 8 petals. One of the points of intersections of the two curves is when theta = pi/8. The segment of the curve r=4 sin (2theta) from 0 to pi/8 gives you half the area of a petal. The total area is 16 times the area of half a petal, so the final answer is 8pi - 16.

7. Find all points of intersection of the curves $r = \sin\theta$ and $r = \sin 2\theta$. Answer: Set the two equations equal to each other. You get 5 values for theta. However, after computing the corresponding r values, you realize that there are really only 3 points of intersection: the pole, (r=sqrt(3)/2, theta=pi/3) in polar coordinates, and (r=-sqrt(3)/2, theta=5pi/3) in polar coordinates. See also similar example, Example 3 page 671.

HW 10.4 #2

Find the area of the shaded region enclosed by the polar curve $r = \sqrt{\theta}$.



Answer:

The curve is an infinite spiral for θ in $[0, \infty)$ but the shaded region is enclosed by the curve $r = \sqrt{\theta}$ from 0 to 2π (and the positive polar axis).

So the area of the shaded region is

$$\int_0^{2\pi} \frac{1}{2}\left(\sqrt{\theta}\right)^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}\theta \, d\theta$$

$$= \frac{1}{2}\frac{\theta^2}{2}\Big|_0^{2\pi}$$

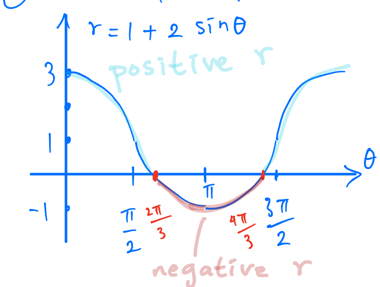
$$= \frac{(2\pi)^2}{4} - 0$$

$$= \boxed{\pi^2}$$

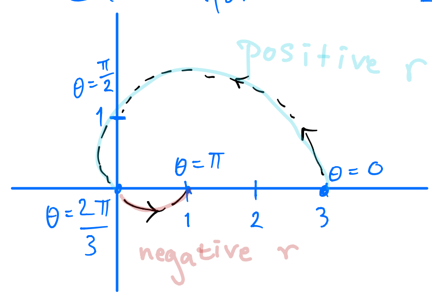
HW 10.4 #5

Find the area of the region inside the larger loop and outside the smaller loop of the polar curve $r = 1 + 2\cos\theta$.

① Sketch curve



Curve for $0 \le \theta \le \pi$:

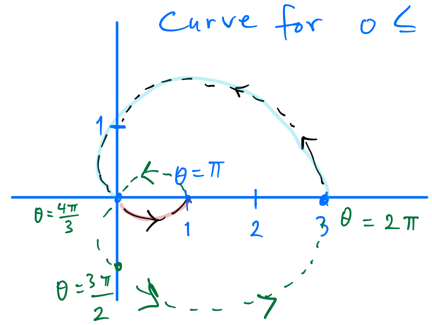


$0 = 1 + 2\cos\theta$

$-\frac{1}{2} = \cos\theta \Rightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

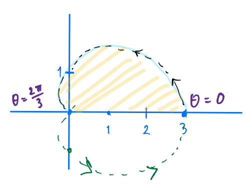
$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

Curve for $0 \le \theta \le 2\pi$:



② Compute area inside the larger loop A_{Larger}

A_{larger} is twice the area of the shaded region



$$A_{Larger} = 2\int_0^{\frac{2\pi}{3}} \frac{1}{2}(1 + 2\cos\theta)^2 \, d\theta$$

$$= \int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + 4\cos^2\theta \, d\theta$$

$$= \int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + \frac{4}{2}(1 + \cos(2\theta)) \, d\theta$$

$$= \boxed{2\pi + \frac{3}{2}\sqrt{3}}$$

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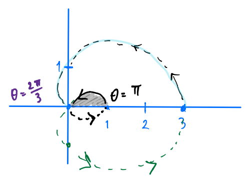
(Con't #5)

③ Compute area inside the smaller loop $A_{Smaller}$

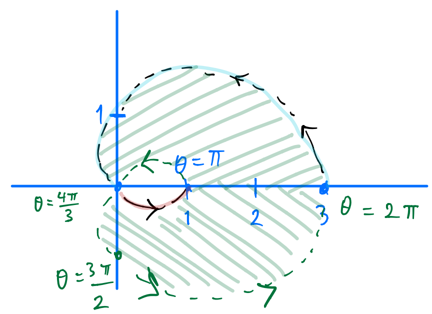
$A_{smaller}$ is twice the area of the shaded region



$$A_{smaller} = 2 \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2} (1 + 2\cos\theta)^2 \, d\theta$$

$$= \boxed{\pi - \frac{3}{2}\sqrt{3}}$$

④ Compute the area of the region inside the larger loop and outside the smaller loop.



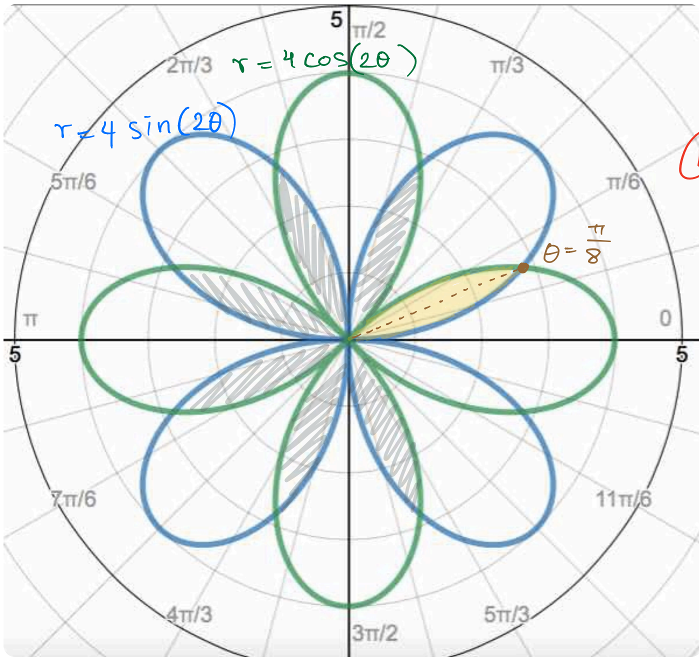
The desired area is the area from step ② minus the area from step ③.

$$A = A_{Larger} - A_{smaller}$$

$$= \left(2\pi + \frac{3}{2}\sqrt{3}\right) - \left(\pi - \frac{3}{2}\sqrt{3}\right)$$

$$= \pi + 3\sqrt{3}$$

——— the end of #5 ———



HW 10.4 #6

Find the area of the region that lies inside both $r = 4\sin 2\theta$ and $r = 4\cos 2\theta$

1. Each curve has 4 petals. The region inside both curves has 8 petals.

We'll use the petal I've shaded in yellow.

Note:

- The high bound of the yellow petal is the (green) curve $r = 4\cos(2\theta)$
- The low bound of the yellow petal is the (blue) curve $r = 4\sin(2\theta)$.

2. Find the intersection point of the blue and green curve around the yellow petal. Note: θ is between 0 and $\frac{\pi}{4}$.

Set $4\sin 2\theta = 4\cos 2\theta \Rightarrow$

$$\frac{\sin 2\theta}{\cos 2\theta} = 1 \Rightarrow$$

$$2\theta = \frac{\pi}{4} \Rightarrow$$

$$\theta = \frac{\pi}{8}$$

3. Compute the area of the upper half of the yellow petal

$$A_{upper} = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}(4\cos 2\theta)^2\, d\theta$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2} 4^2 \left(\frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{2}(\pi - 2)$$

3'. Alternatively, compute the area of the lower half of the yellow petal, $A_{lower} = \int_0^{\frac{\pi}{8}} \frac{1}{2}(4\sin 2\theta)^2\, d\theta$

$$= \int_0^{\frac{\pi}{8}} \frac{1}{2} 4^2 \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{2}(\pi - 2)$$

4. The area of the yellow petal is $\pi - 2$. Multiply this by 8 to get the total area, $\boxed{8\pi - 16}$