Sec 11.10 Part 4

Multiplication and Division of Power Series

If power series are added or subtracted, they behave like polynomials (Theorem 11.2.8 shows this). In fact, as the following example illustrates, they can also be multiplied and divided like polynomials. We find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

EXAMPLE 13 Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

SOLUTION

(a) Using the Maclaurin series for e^x and $\sin x$ in Table 1, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\begin{array}{rl}
 & 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots \\
\times & x \qquad\qquad - \tfrac{1}{6}x^3 + \cdots \\
\hline
 & x + x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \cdots \\
+ & - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^4 - \cdots \\
\hline
 & x + x^2 + \tfrac{1}{3}x^3 + \cdots
\end{array}$$

Thus
$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in Table 1, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

We use a procedure like long division:

$$\begin{array}{r|l}
 & x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\
\hline
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots & x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots \\
 & x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \cdots \\
\hline
 & \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\
 & \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\
\hline
 & \tfrac{2}{15}x^5 + \cdots
\end{array}$$

Thus
$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$ ■

Although we have not attempted to justify the formal manipulations used in Example 13, they are legitimate. There is a theorem which states that if both $f(x) = \Sigma\, c_n x^n$ and $g(x) = \Sigma\, b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division we require $b_0 \neq 0$; the resulting series converges for sufficiently small $|x|$.

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Relevant Exercises: # 67 – 72 pg 772.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

#67, 71, 70, 72

Find the first three nonzero terms in the Maclaurin series for

#67 $f(x) = e^{-x^2} \cos x$. #71 $(\arctan x)^2$. #70 $e^x \ln(1+x)$. #72 $e^x (\sin x)^2$

Method 1:

method 2:

#68-69 Find the first three nonzero terms in the Maclaurin series for

#68 $f(x) = \sec x$

#69 $f(x) = \dfrac{x}{\sin x}$

Method 1: Compute $f(0), f'(0), \dfrac{f''(0)}{2!}$

$\boxed{\text{Method 2}}$: Use division of power series and Table 1.

#67, 71, 70, 72

Find the first three nonzero terms in the Maclaurin series for

#67 $f(x) = e^{-x^2}\cos x$. #71 $(\arctan x)^2$. #70 $e^x \ln(1+x)$. #72 $e^x(\sin x)^2$

in class

HW

Method 1: Compute $f(0)$, $f'(0)$, $\frac{f''(0)}{2!}$

$\boxed{\text{method 2}}$: Use multiplication of power series and Table 1.

Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ from Table 1,

we have $e^{-x^2} = 1 + \frac{-x^2}{1!} + \frac{x^4}{2!} + \frac{-x^6}{3!} + \cdots$

Table 1 also gives us

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Multiply these two power series:

$$\begin{array}{r} 1 - x^2 + \frac{1}{2}x^4 + \cdots \\ 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \\ \hline \times \end{array}$$

$$\begin{array}{l} 1 - x^2 + \frac{1}{2}x^4 + \cdots \\ \quad -\frac{1}{2}x^2 - \frac{1}{2}x^2(-x^2) - \frac{1}{2}x^2\left(\frac{1}{2}x^4\right) + \cdots \\ \qquad \frac{1}{24}x^4 + \frac{1}{24}x^4(-x^2) + \frac{1}{24}x^4\left(\frac{1}{2}x^4\right) + \cdots \\ \hline + \end{array}$$

$$1 - \frac{3}{2}x^2 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{24}\right)x^4 + \cdots$$

The first three nonzero terms

#68-69 Find the first three nonzero terms in the Maclaurin series for

#68 $f(x) = \sec x$ in class

#69 $f(x) = \dfrac{x}{\sin x}$

HW

Method 1: Compute $f(0), f'(0), \dfrac{f''(0)}{2!}$

Method 2: Use division of power series and Table 1.

Since $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$$

Do "long division":

$$1 + \frac{1}{2}x^2 + \left(\frac{6-1}{24}\right)x^4 + \dots\dots$$

$$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \;\big)\; 1$$

$$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \quad -$$

$$\frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \dots$$

$$\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{48}x^6 - \dots \quad -$$

$$\left(-\frac{1}{24} + \frac{1}{4}\right)x^4 + \left(\frac{1}{720} - \frac{1}{48}\right)x^6 + \dots$$

can keep going

The first 3 nonzero terms of the Maclaurin series for $\sec x$ is

$$1, \quad \frac{1}{2}x^2, \quad \frac{5}{24}x^4.$$