Contents

1 11.10 Taylor series

Khan Academy Taylor, Maclaurin, and Power series online quizzes (click):
`https://www.khanacademy.org/math/calculus-home/series-calc/challenge-exercises-series-calc/e/taylor-maclaurin-power-series-challenge`
Khan Academy Series estimation online quizzes (click):
`https://www.khanacademy.org/math/calculus-home/series-calc/challenge-exercises-series-calc/e/series-estimation-challenge`

1. i. *If* f has a power series representation at 4, that is, if $f(x) = \sum_{n=0}^{\infty} c_n(x-4)^n$ for $|x-4| < R$, then its coefficients are given by the formula $c_n =$ __________________.

Solution: Theorem 5 on page 760.

ii. Circle all the true statements and cross out all the false statements, and justify.

(a) If the series $\sum_{n=1}^{\infty} c_n x^n$ converges for $|x| < R$, then $\lim_{n\to\infty} c_n x^n = 0$ for $|x| < R$.

Solution: Answer: True. See explanation, first sentence of pg 763. Or see Sec 11.2, Thm 6, pg 713.

(b) If the series $\sum_{n=1}^{\infty} c_n x^n$ diverges for $x = 5$, then $\lim_{n\to\infty} c_n x^n \neq 0$ for $x = 5$.

Solution: Answer: False. A counterexample: $c_n = \frac{1}{n\ 5^n}$. See Ex. 9 Sec 11.2, pg 713.

iii. Find the Maclaurin series for $f(x) = 6(1-x)^{-2}$. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence.

Solution: The Maclaurin series is $\boxed{\sum_{n=0}^{\infty} 6(n+1)x^n}$.

Option 1 (Sec 11.10): Use Taylor series theorem/formulas 5,6,7 on pg 760. Follow Example 8 but replace $(1+x)^k$ with $(1-x)^{-2}$.

Option 2 (Sec 11.9): First find the power series of $(1-x)^{-1}$ using geometric series. Then use term-by-term differentiation to find the power series for $-(1-x)^{-2}$.

The radius of convergence is $\boxed{R = 1}$ by Ratio Test (if you use Option 1) or by geometric series theorem plus term-by-term differentiation theroem.

iv. Use a Maclaurin series given in this table `http://egunawan.github.io/fall18/exams/exam3practice/11_10_table01.pdf` (will be given) to obtain the Maclaurin series for the function $f(x) = 8e^x + e^{8x}$. Find the radius of convergence.

Solution: Answer: Use the table to get $e^x = \sum_{n=1}^{\infty} x^n n!$. Apply the Composition Theorem with $h(x) = 8x$ and $f(t) = e^t$ to get $e^{8x} = \sum_{n=1}^{\infty} \frac{(8x)^n}{n!}$. Apply 'sum' theorem for series (pg 714 Sec 11.2) to get the sum $\boxed{\sum_{n=0}^{\infty} (8 + 8^n) \frac{x^n}{n!}. \text{ The series is convergent for all real numbers}}$.

v. Evaluate the indefinite integral $\left(8 \int \frac{e^x - 1}{5x} \text{ dx}\right)$ as an infinite series.

Solution: Answer: Read Example 11 pg 768-769 for similar problem. This answer gives the Maclaurin series but you can choose a different Taylor series centered not at 0. First either use the table or directly evaluate the Maclaurin series for $e^x - 1 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$. Multiply this Maclaurin series by $\frac{1}{x}$ to get $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$. Apply term-by-term integration to get final answer, $\boxed{\frac{8}{5} \sum_{n=1}^{\infty} \frac{x^n}{(n)n!} + C}$.

vi. Find the Maclaurin series for $f(x) = e^{-4x}$ using the definition of a Maclaurin series. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence R.

Solution: Answer: Follow Example 1 pg 760 but replace x with $-4x$. You get $\boxed{e^{-4x} = \sum_{n=0}^{\infty} \left(\frac{(-4)^n}{n!}\right) x^n. \text{ The series is convergent for all real numbers}}$.

2. (a) Use Table 1 to show that $\frac{d}{dx} \cos(x) = -\sin(x)$.

(b) Write the first 3 nonzero terms of the Maclaurin series for $\tan(x)$ using Table 1 and long division of power series.

Solution: Example 13 Sec 11.10 page 770.

(c) Use the series that you just computed for $\tan(x)$ to evaluate

$$\lim_{x \to 0} \frac{\tan(x) - x}{x^3}.$$

Solution: Answer: $\boxed{1/3}$.

(d) Use a different method to evaluate $\lim_{x \to 0} \frac{\tan(x) - x}{x^3}$.

Solution: answer: $\boxed{1/3}$.

(e) Write the first 3 nonzero terms of the Maclaurin series for $e^x \sin(x)$ using Table 1 and multiplication of power series.

Solution: answer: Follow Example 13 Sec 11.10 page 770.

(f) Write the first 3 nonzero terms of the Maclaurin series for $\sec(x)$ using long division of power series and Table 1.

Solution: answer: $\boxed{1 + x^2/2 + 4x^4/24}$

3. (a) An application of the Alternating Series Estimation Theorem is a way to ensure that we can get an approximation to the *definite* integral $\int_0^1 e^{-x^2}$ dx using series so that the approximation is within a certain error bound (for example 1/1000).

T **F**

Solution: answer: True. See Example 11 Sec 11.10 pg 769.

(b) We can always use the Alternating Series Estimation Theorem to ensure that we can get an approximation of a *function* using its Taylor polynomial so that the approximation is within a certain error bound (for example, 1/1000) on a certain interval. **T** **F**

Solution: False. This only works when the resulting series (for every x in the domain of the function) is alternating. See Example 1 Sec 11.11 pg 775.

4. (a) True or False? If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \ldots$ for $|x| < 1$ then $f'''(0) = 30$.

Solution: True. The coefficient for x^n is equal to $\dfrac{f(n)(0)}{n!}$ by definition of Maclaurin series. Since $5 = \dfrac{f'''(0)}{3!}$, we have $f'''(0) = 30$.

(b) Can you write a Maclaurin series for $f(x) = \sqrt[3]{x}$? Explain why or why not.

Solution: answer: The function $f(x) = \sqrt[3]{x}$ is not differentiable at 0, so we cannot define a Maclaurin series for $f(x)$.

5. Using Table 1 (series), prove that $Re^{i\theta} = R\cos\theta + iR\sin\theta$

Solution: `https://egunawan.github.io/fall18/lecture_notes_f18/f18_week12_day2_notes10_3part1.pdf`

6. True or False, with justification.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ **T** **F**

Solution: answer: True because $e^{-1} = \frac{1}{e}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = -e$ **T** **F**

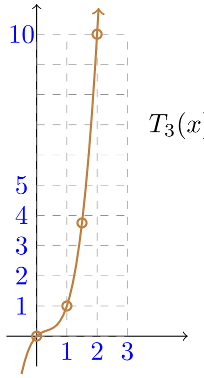
Solution: answer: False. The left hand side is equal to e^{-1}.

(c) If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \ldots$ converges for all x, then $f^{'''}(0) = 2$ **T** **F**

Solution: True. The coefficient for x^n is equal to $\dfrac{f(n)(0)}{n!}$ by definition of Maclaurin series. Since $\dfrac{1}{3} = \dfrac{f'''(0)}{3!}$, we have $f'''(0) = 2$.

7. (a) Find the 3rd degree Taylor polynomial of the function $f(x) = xe^{-2x}$ centered at 0. Sketch this polynomial. Label all the important points - label at least four convenient points.

Solution: answer: $\boxed{T_3(x) = 2x^3 - 2x^2 + x}$



$T_3(x) = 2x^3 - 2x^2 + x$

(b) (i) Approximate $f(x) = \ln(1+2x)$ by a Taylor polynomial of degree 3 centered at 1.

Solution: answer: $\boxed{\ln 3 + 2/3(x-1) - 2/9(x-1)^2 + 8/81(x-1)^3}$.

(ii) Use Taylor's Inequality to estimate the accuracy of the approximation $T_3(x)$ of $f(x)$ when $0.8 \le x \le 1.2$. You do not need to simplify your answer.

Solution: answer:
Note that $f^{(4)}(x) = -96/(1+2x)^4$
$|R_3(x)| \le \frac{M}{4!}|x-1|^4$, where $|f^{(4)}(x)| \le M$. $0.8 \le x \le 1.2$ implies $-0.2 \le x-1 \le 0.2$ which implies $|x-1| \le 0.2$ which implies $|x-1|^4 \le 0.2^4 = 0.0016$. The largest possible value for $|f^{(4)}(x)| = 96/(1+2x)^4$ in the interval is when $x = 0.8$, so we let $\boxed{M = |f^{(4)}(0.8)| = 96/(2.6)^4}$. So the error is within $\boxed{\frac{M}{4!}0.0016}$. You don't need to simplify M.

(c) Approximate $f(x) = e^{4x^2}$ by a Taylor polynomial with degree 3 centered at 0.

Solution: answer: $\boxed{1 + 0x + 4x^2 + 0x^3 = 1 + 4x^2}$

8. Compute the Taylor series for $f(x) = \ln(x)$ at $a = 10$.

Solution: *Thinking about the problem:* We will differentiate $\ln x$ enough times to see a pattern. The pattern will give us the coefficients in the Taylor series.

Doing the problem:

The first several higher derivatives of $f(x) = \ln x$ are in the table below.

n	0	1	2	3	4	5	6	7
$f^{(n)}(x)$	$\ln x$	$1/x$	$-1/x^2$	$2/x^3$	$-6/x^4$	$24/x^5$	$-120/x^6$	$720/x^7$

The pattern for $n \ge 1$ is $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$, so the Taylor series of $\ln x$ at $a = 10$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n &= f(10) + \sum_{n=1}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{10^n n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-10)^n}{10^n n} \\
&= \ln 10 + \frac{x-10}{10} - \frac{(x-10)^2}{200} + \frac{(x-10)^3}{3000} - \frac{(x-10)^4}{40000} + \cdots
\end{aligned}$$

9. Find the Taylor series for $f(x) = \sqrt{x}$ centered at 9.

Solution: `https://www.overleaf.com/read/krtzsqgykktb`

10. Determine the 2nd-degree Taylor polynomial $T_2(x)$ for $\arctan x$ at $a = 1$ and use Taylor's inequality to bound $|R_2(x)|$ if $|x-1| \le \frac{1}{2}$, where $\arctan x = T_2(x) + R_2(x)$.

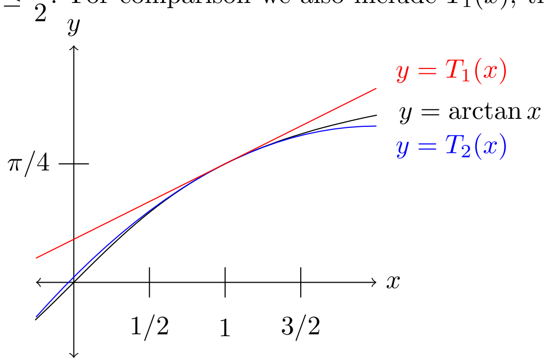
Solution:
Thinking about the problem: The 2nd-degree Taylor polynomial for a function $f(x)$ at $a = 1$ is

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2.$$

We will find the coefficients when $f(x) = \arctan x$. To bound $R_2(x)$ when $|x-1| \le \frac{1}{2}$ with Taylor's inequality, we need an M such that $|f'''(x)| \le M$ for $|x-1| \le \frac{1}{2}$.
Doing the problem:
To find $T_2(x)$, here is a table of higher derivatives of $f(x) = \arctan x$.

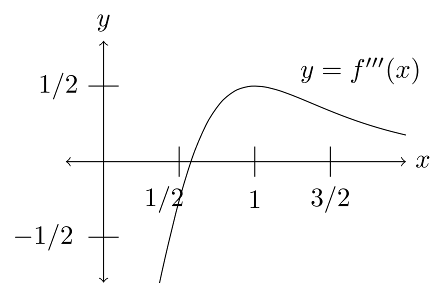
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\arctan x$	$\frac{\pi}{4}$
1	$\frac{1}{1+x^2}$	$\frac{1}{2}$
2	$\frac{-2x}{(1+x^2)^2}$	$-\frac{1}{2}$

From the table, $T_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1/2}{2}(x-1)^2 = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$. The graphs below show $T_2(x)$ is a good approximation of $\arctan x$ for $|x-1| \le \frac{1}{2}$. For comparison we also include $T_1(x)$, the linear approximation to $\arctan x$ at $a = 1$.



To bound $|R_2(x)|$ for $|x-1| \le 1/2$, we need to a number M such that $|f'''(x)| \le M$ for $|x-1| \le 1/2$. What is the biggest value of $|f'''(x)|$ for $|x-1| \le 1/2$?

From the formula for $f''(x)$ in the table, $f'''(x) = \frac{6x^2-2}{(1+x^2)^3}$. Here is the graph of $f'''(x)$.



There is a local maximum of $f'''(x)$ at $x = 1$ where $f'''(1) = 1/2$ (the 4th derivative $f^{(4)}(x) = 24x(1 - x^2)/(1 + x^2)^4$ vanishes at $x = 1$) and at endpoints $f'''(1/2) \approx -.256$, and $f'''(3/2) \approx .335$, so $-.256 \leq f'''(x) \leq 1/2$ when $|x - 1| \leq 1/2$. So use $M = |f'''(1)| = 1/2$:

$$|x-1| \leq \frac{1}{2} \Longrightarrow |R_2(x)| \leq \frac{M}{3!}|x-1|^3 = \frac{1}{12}|x-1|^3 \leq \frac{1}{12}\left(\frac{1}{2}\right)^3 = \frac{1}{12 \cdot 8} = \frac{1}{96} \approx .0104.$$

2 10.1-10.2 Calculus with parametric equations

11. For the following two parametric curves

$$(1)\ x = \cos t,\ y = \sin t \text{ for } 0 \leq t \leq 2\pi, \quad (2)\ x = -\sin(2t),\ y = -\cos(2t) \qquad \text{for} \qquad 0 \leq t \leq \frac{3\pi}{2}$$

eliminate the parameter to obtain an equation for the curve that directly relates x and y (non-parametric form of the curve) and then sketch the curve with an arrow indicating the direction it is traced out as t increases. Find the initial and final points.

Solution:

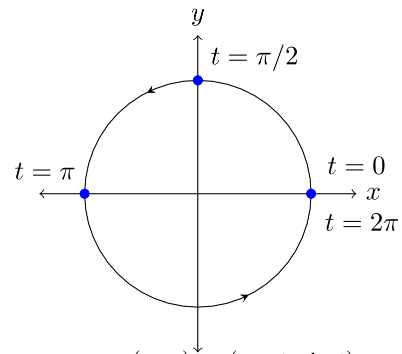
Thinking about the problem: Since x and y are essentially sines and cosines (or *vice versa*) of the same value (t or $2t$), we expect the equation for the curve directly relating x and y will be a circle of radius 1. The parametric formulas for x and y will tell us the initial and final points of the traced circle, the direction the circle is traced out, and how many times the circle is traced out.

Doing the problem:

(1) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$: the curve $(x(t), y(t)) = (\cos t, \sin t)$ is part of the unit circle. In the table below we compute $(x(t), y(t))$ at $t = 0$, $t = \pi/2$, $t = \pi$, and $t = 2\pi$.

t	0	$\pi/2$	π	2π
$(\cos t, \sin t)$	$(1, 0)$	$(0, 1)$	$(-1, 0)$	$(1, 0)$

These points are marked in the figure below, which shows the curve traced out by the parametrization is a circle going counterclockwise once. The direction the curve is traced out is indicated with arrows. It starts at $(x(0), y(0)) = (1, 0)$ and ends at $(x(2\pi), y(2\pi)) = (1, 0)$.

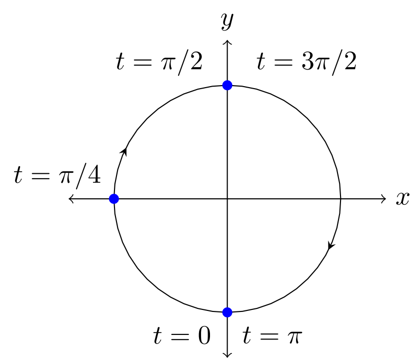


$(x, y) = (\cos t, \sin t)$

(2) $x^2 + y^2 = (-\sin(2t))^2 + (-\cos(2t))^2 = \sin^2(2t) + \cos^2(2t) = 1$, so the curve $(x(t), y(t)) = (-\sin(2t), -\cos(2t))$ traces out part of the unit circle. The table below shows $(x(t), y(t))$ at $t = 0$, $t = \pi/4$, $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$.

t	0	$\pi/4$	$\pi/2$	π	$3\pi/2$
$(-\sin(2t), -\cos(2t))$	$(0, -1)$	$(-1, 0)$	$(0, 1)$	$(0, -1)$	$(0, 1)$

These points are marked in the figure below, which shows the curve traced out by the parametrization is a circle going clockwise *one and a half times*. The direction of increasing t is indicated with arrows. It starts at $(x(0), y(0)) = (0, -1)$ and ends at $(x(3\pi/2), y(3\pi/2)) = (0, 1)$.



$(x, y) = (-\sin(2t), -\cos(2t))$

12. i. (a) Find *parametric* equations for the top half of the circle centered at $(2, 3)$ with radius 5, oriented *clockwise*.

Solution: Answer:

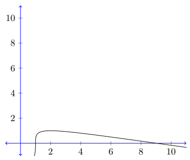
$$x = 2 + 5\cos(-t)$$
$$y = 3 + 5\sin(-t)$$
$$\text{for } \pi \le t \le 2\pi$$

(b) Eliminate the parameter to find a Cartesian equation of the curve.

ii. Consider the curve described by the parametric equations

$$x = t^3 + 1$$
$$y = 2t - t^2, \quad \text{for } -\infty < t < \infty$$

(a) Mark the orientation on the curve (direction of increasing values of t).



(b) Find the area enclosed by the x-axis and the given curve.

Solution: Answer:

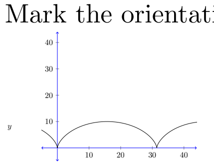
$$\begin{aligned}\int_1^9 y \text{ dx} &= \int_{t=0}^{t=2} y(t)x'(t)dt \\ &= \int_0^2 (2t - t^2)(3t^2)dt \\ &= \int_0^2 (6t^3 - 3t^4)dt \\ &= \frac{6}{4}t^4 - \frac{3}{5}t^5\Big|_0^2 \\ &= \frac{3(16)}{2} - \frac{3(32)}{5} \\ &= \frac{3(40-32)}{5} \\ &= \boxed{\frac{24}{5}}\end{aligned}$$

(c) Perform and describe a reality check by comparing your answer and the graph which has been drawn to scale.

iii. Consider the cycloid which is described by the parametric equations

$$\begin{aligned} x &= 5(t - \sin t) \\ y &= 5(1 - \cos t), \qquad \text{for } \infty < t < \infty \end{aligned}$$

(a) Mark the orientation on the curve (direction of increasing values of t).

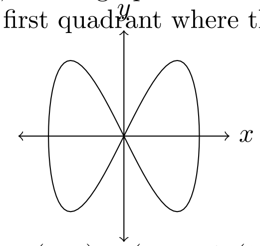


(b) Find the area enclosed by the x-axis and *one* arch of the cycloid. Hint: $dx = 5(1 - \cos t)\ dt$.

Solution: Answer: $3\pi 5^2$

(c) Perform a reality check by comparing your answer and the graph (which is drawn to scale).

13. On the parametric curve $(x, y) = (\cos(t), \sin(2t))$, whose graph is below, determine (a) the slopes of the two tangent lines at the origin and (b) coordinates of the point in the first quadrant where the tangent line has slope -2.



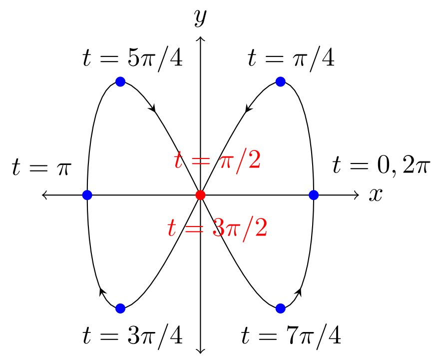
$(x, y) = (\cos t, \sin(2t))$

Solution:

Thinking about the problem: We will figure out t-values where $(x(t), y(t))$ is the origin and compute $\frac{dy}{dx}$ at such t. To figure out where $\frac{dy}{dx} = -2$ in the first quadrant, rewrite this as $\frac{dy}{dt} = -2\frac{dx}{dt}$, find the t-value where that happens in the first quadrant, and then compute $(x(t), y(t))$ to get coordinates.

Doing the problem:

Below we mark points at t-values that are multiples of $\pi/4$ from 0 to 2π. Starting at $(1,0)$ where $t=0$, the curve is traced out through quadrants 1, 3, 2, and 4 (note the arrows) before returning to $(1,0)$.



(a) The derivative on this curve is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(2t)}{-\sin t} = -2\frac{\cos(2t)}{\sin t}$. The two times the curve passes through the origin are at $t=\pi/2$ and $t=3\pi/2$, and the derivatives at these t-values are

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = -2\frac{\cos(2(\pi/2))}{\sin(\pi/2)} = -2\frac{\cos(\pi)}{\sin(\pi/2)} = -2\left(\frac{-1}{1}\right) = \boxed{2}$$

and

$$\left.\frac{dy}{dx}\right|_{t=3\pi/2} = -2\frac{\cos(2(3\pi/2))}{\sin(3\pi/2)} = -2\frac{\cos(3\pi)}{\sin(3\pi/2)} = -2\left(\frac{-1}{-1}\right) = \boxed{-2}.$$

(b) To find where $\frac{dy}{dx} = -2$ in the first quadrant we will solve $-2\frac{\cos(2t)}{\sin t} = -2$ with $0 \le t \le \pi/2$. That means $\cos(2t) = \sin t$, or $1 - 2\sin^2(t) = \sin t$ by the double-angle formula for $\cos(2t)$. By the quadratic formula, $1 - 2a^2 = a$ at $a = 1/2$ and -1, so we want to solve $\sin t = 1/2$ and $\sin t = -1$. When $0 \le t \le \pi/2$ the number $\sin t$ is not negative so we just need to solve $\sin t = 1/2$ and that happens at $t = \pi/6$. Thus the point in quadrant 1 with tangent slope -2 is $(x(\pi/6), y(\pi/6)) = (\cos(\pi/6), \sin(\pi/3)) = \boxed{(\sqrt{3}/2, \sqrt{3}/2)}$.

3 10.3-10.4 Polar equations, sketch, derivatives and area

14. (a) Sketch the polar equation $r = \frac{5}{2}$

(b) Sketch the polar equation $\theta = \frac{\pi}{4}$

(c) Convert the polar equation $r = 3$ to Cartesian.

Solution: Answer: $x^2 + y^2 = 9$

(d) Convert the polar equation $\theta = \frac{\pi}{3}$.

Solution: Answer: $y = \sqrt{3}x$

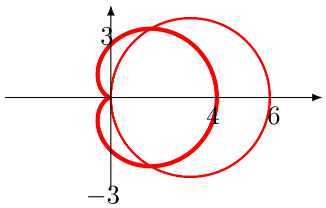
(e) Convert the polar equation $\theta = \frac{\pi}{6}$.

Solution: Answer: $y = \frac{\sqrt{3}}{3}x$

(f) Convert the polar equations $r = 9\cos\theta$ to Cartesian.

Solution: Answer: $(x-4.5)^2+y^2=(4.5)^2$

15. Consider the circle $r=6\cos\theta$ and the cardioid $r=2+2\cos\theta$.



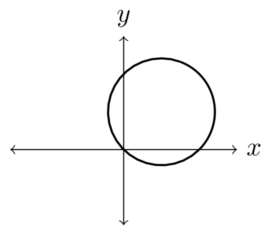
(a) Mark points on *both* curves where $\theta=0, \frac{\pi}{4}$, and $\frac{\pi}{2}$.

(b) Shade in the area inside the circle and outside the cardioid.

(c) Find the area (which you shade) inside the circle and outside the cardioid.

Solution: Answer: $\int_0^{\pi/3} 2(8\cos^2\theta-1-2\cos\theta)d\theta=4\pi$.

16. Below is the plot of the polar equation $r=\sin\theta+\cos\theta$.



Fill in the table below, use it to determine the orientation of the curve (direction of increasing θ), and find the equation of the tangent line to the curve at $(x,y)=(0,0)$.

θ	$\sin\theta+\cos\theta$	(r,θ)	(x,y)
0			
$\pi/4$			
$\pi/2$			
π			

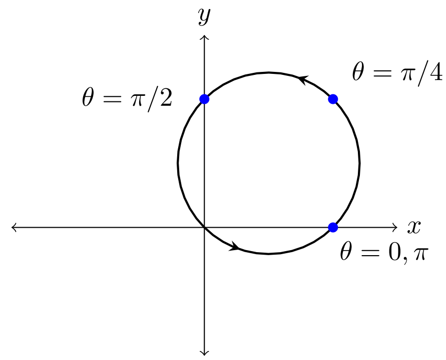
Solution:

Thinking about the problem: We can fill in the table using the polar equation and the conversion formulas from polar to Cartesian coordinates. To find the tangent line at $(x,y)=(0,0)$ we will find the value of θ at which the curve passes through the origin and then compute dy/dx at that value of θ.

Doing the problem:

Using the formulas $x=r\cos\theta$ and $y=\sin\theta$ we find the following:

θ	$\sin\theta+\cos\theta$	(r,θ)	(x,y)
0	1	$(1,0)$	$(1,0)$
$\pi/4$	$\sqrt{2}$	$(\sqrt{2},\pi/4)$	$(1,1)$
$\pi/2$	1	$(1,\pi/2)$	$(0,1)$
π	-1	$(-1,\pi)$	$(1,0)$



From the marked points already put on the circle, it is natural to guess $\theta = 3\pi/4$ will correspond to the origin, and indeed for this value of θ we have $\sin\theta = 1/\sqrt{2}$ and $\cos\theta = -1/\sqrt{2}$, so $r = 1/\sqrt{2} - 1/\sqrt{2} = 0$ and thus the point is the origin.

To find the equation of the tangent line, we compute $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ at $\theta = 3\pi/4$ and then take the ratio to get $\frac{dy}{dx}$. First, since $y = r\sin\theta$ and $r = \sin\theta + \cos\theta$,

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = (\cos\theta - \sin\theta)\sin\theta + r\cos\theta = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 = -1$$

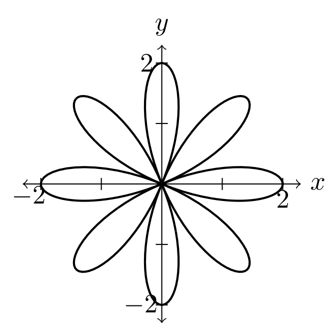
and since $x = r\cos\theta$ and $r = \sin\theta + \cos\theta$,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta + r(-\sin\theta) = (\cos\theta - \sin\theta)\cos\theta - r\sin\theta = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + 0 = 1.$$

Therefore $\left.\frac{dy}{dx}\right|_{\theta=3\pi/4} = \frac{-1}{1} = -1.$

Therefore the tangent line through the origin has slope -1, so its equation is $y = -x$.

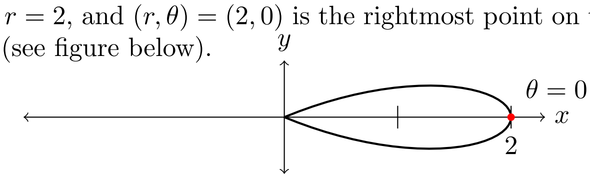
17. Below is a graph of $r = 2\cos 4\theta$. Determine the area enclosed by it.



Solution: *Thinking about the problem:* The graph has 8 petals, all with the same area, so the total area is 8 times the area of one petal. We will compute the area of one petal using the polar region area formula.

Doing the problem:

On the curve, when $\theta = 0$ we have $r = 2$, and $(r, \theta) = (2, 0)$ is the rightmost point on the petal crossing the positive x-axis. We will find the area of this petal (see figure below).



The graph is at the origin for the first time with $\theta > 0$ when $2\cos(4\theta) = 0$ for the smallest $\theta > 0$. That means $4\theta = \pi/2$, so $\theta = \pi/8$. By symmetry, the petal containing $(2, 0)$ is traced out for the continuous range of angles $-\pi/8 \le \theta \le \pi/8$. (While $-\pi/8 = 15\pi/8$ as polar angles, the graph for $\pi/8 \le \theta \le 15\pi/8$ is **not** the petal above, but all the others! Do you see why?) The area of the petal above is therefore

$$\begin{aligned}
\int_{-\pi/8}^{\pi/8} \frac{1}{2}(2\cos 4\theta)^2\, d\theta &= \int_0^{\pi/8} (2\cos 4\theta)^2\, d\theta \quad \text{since } \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ for even } f(x)\\
&= \int_0^{\pi/8} 4\cos^2(4\theta)\, d\theta\\
&= \int_0^{\pi/8} 4\cdot\frac{1+\cos(8\theta)}{2}\, d\theta \quad \text{since } \cos^2 x = \frac{1+\cos(2x)}{2}\\
&= \int_0^{\pi/8} (2 + 2\cos(8\theta))\, d\theta\\
&= \left(2\theta + \frac{1}{4}\sin(8\theta)\right)\Big|_0^{\pi/8}\\
&= \left(\frac{2\pi}{8} + \frac{1}{4}\sin(\pi)\right) - \left(0 + \frac{1}{4}\sin(0)\right)\\
&= \frac{\pi}{4} \quad \text{since } \sin(\pi) = 0.
\end{aligned}$$

Therefore the area enclosed by the whole graph (8 petals) is $8(\pi/4) = \boxed{2\pi}$.

4 9.1 Modeling with differential equations

18. (a) For what values of k does the function $y = \cos(kt)$ satisfy the differential equation $4y'' = -9y$?

Solution: Answer: $k = -\frac{3}{2}$, $k = \frac{3}{2}$

(b) Circle all functions which are solutions to $4y'' = -9y$. (Possibly none or all).

1. $y = -\cos(\frac{3t}{2})$

Solution: Answer: Yes

2. $y = \cos(\frac{3t}{2}) + 1$

Solution: Answer: No

3. $y = \sin(\frac{3t}{2})$

Solution: Answer: Yes

4. $y = \sin(\frac{3t}{2}) + \cos(\frac{3t}{2})$

Solution: Answer: Yes

(c) True or false? Every member of the family of functions $y = \dfrac{4\ln(x) + C}{x}$ is a solution of the differential equation

$$x^2y' + xy = 4$$

Solution: Answer: True. Show this by substituting y and y' into the differential equation.

(d) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(1) = 2$.

Solution: Answer: $y = \frac{4\ln(x)+2}{x}$.

(e) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(2) = 1$.

Solution: Answer: $y = \frac{4\ln(x)+2-4\ln(2)}{x}$.

(f) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(3) = 1$.

Solution: Answer: $y = \frac{4\ln(x)+3-4\ln(3)}{x}$.

(g) What can you say about a solution of the differential equation $y' = -\frac{1}{2}y^2$ *just* by looking at the differential equation? Circle all possibilities.

1. The function y must be equal to 0 on any interval on which it is defined.

   Solution: Answer: no.

2. The function y must be strictly increasing on any interval on which it is defined.

   Solution: Answer: no.

3. The function y must be increasing (or equal to 0) on any interval on which it is defined.

   Solution: Answer: no.

4. The function y must be decreasing (or equal to 0) on any interval on which it is defined.

   Solution: Answer: correct.

5. The function y must be strictly decreasing on any interval on which it is defined.

   Solution: Answer: no.

(h) Verify that all members of the family $y = \frac{2}{x+C}$ are solutions of the differential equation $y' = -\frac{1}{2}y^2$.

(i) Write a solution of the differential equation $y' = -\frac{1}{2}y^2$ that is not a member of the family $y = \frac{2}{x+C}$.

Solution: Answer: $y = 0$

(j) Find a solution of the initial-value problem. $y' = -\frac{1}{2}y^2 \qquad y(0) = 0.1$

Solution: Answer: $\frac{2}{x+20}$

(k) Find a solution of the initial-value problem. $y' = -\frac{1}{4}y^2 \qquad y(0) = 0.2$

Solution: Answer: $\frac{4}{x+20}$

(l) Find a solution of the initial-value problem. $y' = -\frac{1}{3}y^2 \qquad y(0) = 0.5$

Solution: Answer: $\frac{3}{x+6}$

(m) Find a solution of the initial-value problem. $y' = -\frac{1}{6}y^2 \qquad y(0) = 0.5$

Solution: Answer: $\frac{6}{x+12}$

(n) A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.1P\left(1 - \frac{P}{4000}\right)$$

1. For what values of P is the population increasing?

Solution: Answer: $(0, 4000)$. Explanation: You need $1 - P/4000 > 0$ and $P > 0$.

2. For what values of P is the population decreasing?

Solution: Answer: $(4000, \infty)$. Explanation: You need $1 - P/4000 < 0$ and $P > 0$.

3. What are the equilibrium solutions?

Solution: Answer: $P = 4000$ and $P = 0$. Explanation: You need $dP/dt = 0$.

(o) A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 8y^3 + 15y^2.$$

1. What are the constant solutions of the equation?

Solution: Answer: $y = 0, y = 3$, and $y = 5$

2. Sketch the polynomial $t^4 - 8t^3 + 15t^2$. In particular, mark the x-intercepts.
3. For what values of y is y increasing?

Solution: Answer: When y is in one of the intervals $(-\infty, 0), (0, 3), (5, \infty)$

4. For what values of y is y decreasing?

Solution: Answer: When y is in the interval $(3, 5)$

19. (a) True or false? Every differential equation has a constant solution. (If T, explain. If F, give a counterexample.)

Solution: Answer: False. There are many possible counterexamples: think of a function $g(y)$ which has no zeros. You can use $\frac{dy}{dx} = g(y)$ as a counterexample.

(b) Consider the differential equation $\frac{dy}{dt} = 5 - 2y$.

i. Find all constant solution/s.

Solution: Answer: $y = 5/2$

ii. Which of the following is a family of solutions? You may need to circle more than one.

$$y(t) = 1 + Ke^{-2t} \qquad y(t) = -Ke^{-2t} \qquad y(t) = \frac{5}{2} + Ke^{-2t} \qquad y(t) = \frac{5}{2} - Ke^{-2t}$$

Solution: Answer: $y(t) = \frac{5}{2} + Ke^{-2t} \qquad y(t) = \frac{5}{2} - Ke^{-2t}$

iii. Which of the functions below satisfy the differential equation $y'' + y = \sin x$?

(a) $y = \sin x$

(b) $y = \cos x$

(c) $y = \frac{1}{2}x \sin x$

(d) $y = -\frac{1}{2}x \cos x$

Solution:
Thinking about the problem: First we will find y' and then y'' for each of the functions and then compute $y'' + y$ in each case to see if we get $\sin x$.
Doing the problem:

(a) $y = \sin x \implies y'' = -\sin x$
$\implies y'' + y = 0,$

(b) $y = \cos x \implies y'' = -\cos x$
$\implies y'' + y = 0,$

(c) $y = \frac{1}{2}x \sin x \implies y'' = \cos x - \frac{1}{2}x \sin x$
$\implies y'' + y = \cos x,$

(d) $y = -\frac{1}{2}x \cos x \implies y'' = \sin x + \frac{1}{2}x \cos x$
$\implies y'' + y = \sin x.$

The only solution to $y'' + y = \sin x$ among the four functions here is (d) $y = -\frac{1}{2}x \cos x$.

20. (a) Draw a rough sketch of a possible solution to the logistic differential equation $\frac{dP}{dt} = 5P\left(1 - \frac{P}{8}\right)$. **You do not need to solve this differential equation to draw a rough sketch**.

Solution: See sketch in Notes Sec 9.1, or watch `https://www.khanacademy.org/math/ap-calculus-bc/bc-diff-equations/bc-logistic-models/e/logistic-differential-equation`

5 9.3 Separable differential equations

21. (a) Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = -9$$

Solution: Answer: $y = -\sqrt{x^2 + 81}$

(b) Find the solution of the differential equation that satisfies the given initial condition.

$$xy' + y = y^2, \quad y(1) = -8$$

Solution:

$$\begin{aligned}
xy' &= y^2 - y \\
x\frac{dy}{dx} &= y^2 - y \\
\text{Rewrite in differential form:} & \\
\frac{dy}{y^2 - y} &= \frac{1}{x}dx \\
\int \frac{dy}{y(y-1)} &= \int \frac{1}{x}dx \\
\int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy &= \int \frac{1}{x}dx \\
\ln(y-1) - \ln(y) &= \ln x + C \\
\ln\left(\frac{y-1}{y}\right) &= \ln x + C \\
\frac{y-1}{y} &= xe^C \\
1 - \frac{1}{y} &= xe^C
\end{aligned}$$

Now solve for y. You can rewrite $D = e^C$ to make it easier to find the constant, or leave e^C as is. Answer: $\boxed{y = \frac{8}{8-9x}}$

(c) Consider the differential equation $(x^2 + 15)y' = xy$.

i. Find all constant solutions.

Solution: Answer: $y = 0$

ii. Find all solutions.

Solution:

$$\begin{aligned}
\int \frac{1}{y}dy &= \int \frac{x}{x^2 + 15}dx \\
\ln(y) &= \frac{1}{2}\ln(x^2 + 15) + C \text{ use u-substitution} \\
y &= (x^2 + 15)^{1/2}K.
\end{aligned}$$

Answer: $y = K\sqrt{x^2 + 15}$

(d) A tank contains 500 L of brine with 15 kg of dissolved salt. Brine having .2 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and is drained from the tank at 10 L/min. How much salt is in the tank after t minutes? After 20 min? In the long run?

Solution:
Thinking about the problem: Let $y(t)$ be the amount of salt in the tank at t min. We need the rate in and rate out of salt in kg/min. The rate in is the concentration of salt (in kg/L) multiplied by the rate of liquid entering the tank (in L/min), and the rate out is the concentration of salt multiplied by the rate of liquid leaving the tank. After finding $\frac{dy}{dt}$, we solve for $y(t)$ and $y(20)$.

Doing the problem: Let $y(t)$ be the amount of kg of salt in the tank at t minutes, so $y(0) = 15$. The problem also says brine with .2 kg/L of salt enters at a rate of 10 L/min and the whole mixture drains from the tank at 10 L/min. The concentration of salt entering the tank at time t is .2 kg/L, so the rate of salt entering the tank at time t is

$$\text{concentration} \cdot \text{rate of liquid entering the tank} = .2\,\frac{\text{kg}}{\text{L}} \cdot 10\,\frac{\text{L}}{\text{min}} = 2\,\frac{\text{kg}}{\text{min}}.$$

The concentration of salt leaving the tank at time t is

$$\frac{\text{amount of salt in tank}}{\text{volume of tank}} = \frac{y(t)\,\text{kg}}{500\,\text{L}},$$

so the rate of salt leaving of the tank at time t is

$$\text{concentration} \cdot \text{rate of liquid leaving the tank} = \frac{y(t)\text{kg}}{500\,\text{L}} \cdot 10\,\frac{\text{L}}{\text{min}} = \frac{y(t)}{50}\,\frac{\text{kg}}{\text{min}}.$$

Therefore, in kg/min,

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = 2 - \frac{y(t)}{50} = \frac{100 - y(t)}{50}$$

The differential equation $\boxed{\frac{dy}{dt} = \frac{100-y}{50}}$ is separable:

$$\frac{dy}{dt} = \frac{100-y}{50} \Longrightarrow \frac{dy}{100-y} = \frac{dt}{50} \Longrightarrow \int \frac{dy}{100-y} = \int \frac{dt}{50} \Longrightarrow -\ln|100-y| = \frac{t}{50} + C.$$

Thus $\ln|100 - y| = -t/50 - C$, so raising e to both sides, we get $100 - y(t) = \pm e^{-C} e^{-t/50}$. Setting $t = 0$ here, $100 - 15 = \pm e^{-C}$, so $100 - y(t) = 85e^{-t/50}$. Thus $y(t) = 100 - 85e^{-t/50}$. This is the number of kilograms of salt in the tank after t minutes. After 20 minutes there is $y(20) = 100 - 85e^{-20/50} \approx 43.02$ kg of salt.

In the long run, the concentration of salt in the tank must match that of incoming brine (.2 kg/L), so the amount of salt in 500 L should tend to (.2 kg/L)(500 L)= 100 kg, which is consistent with $y(t) \to 100$ as $t \to \infty$.

(e) The differential equation below models the temperature of a 86^o C cup of coffee in a 20^o C room, where it is known that the coffee cools at a rate of 1^o C per minute when its temperature is 70^o C. Solve the differential equation to find an expression for the temperature of the coffee at time t. (Let y be the temperature of the cup of coffee in ^oC, and let t be the time in minutes, with $t = 0$ corresponding to the time when the temperature was 86^o C.)

$$\frac{dy}{dt} = -\frac{1}{50}(y - 20)$$

Solution: Answer: $y = Ke^{-t/50} + 20$. After considering the initial condition, we see that the temperature of the coffee at the time is described by $y = 66e^{-t/50} + 20$.

(f) A tank contains 8000 L of brine with 14 kg of dissolved salt. Pure water enters the tank at a rate of 80 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

1. How much salt is in the tank after t minutes?

   Solution: Answer: $y = 14e^{-t/100}$ kg

2. How much salt is in the tank after 20 minutes?

   Solution: Answer: $14e^{-0.2}$ kg. (Around 11.5 kg). You don't need to approximate.

(g) Find the orthogonal trajectories of the family of curves $y^2 = 8kx^3$. Sketch these orthogonal trajectories.

Solution: Answer: $2x^2 + 3y^2 = C$, a certain family of ellipses.