1 4.4 Limit laws and L'Hospital's Rule

1. i.) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   Solution: Answer: $\boxed{\text{Not enough information}}$. For example, $\lim_{x\to\infty} x - x = 0$, $\lim_{x\to\infty} (x+1) - x = 1$, $\lim_{x\to\infty} x^2 - x = \infty$.

   ii.) If $\lim_{x\to\infty} f(x) = 1$, then $\lim_{x\to\infty} (f(x))^x$ is ...
   (a) zero (b) ∞ (c) 1 (d) another method is needed to determine this.

   Solution: Answer: $\boxed{\text{not enough information}}$. For example, $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$ and $\lim_{x\to\infty} 1^x = 1$.

   iii.) Evaluate $\lim_{x\to 1} \frac{\ln x}{x-1}$ and $\lim_{x\to\infty} \frac{e^x}{x^2}$.

   Solution: $\boxed{1}$ and $\boxed{\infty}$. Sec 4.4 Example 1, pg 306 and Sec 4.4 Example 2, pg 306.

   iv.) Evaluate $\lim_{x\to 0^+} x(\ln x)^3$. Note: This shows up frequently when we compute our improper integral examples.

   Solution: $\boxed{0}$. Similar to Sec 4.4 Example 6, pg 308, but we apply L'hospital's Rule three times.

2 7.1 Integration by Parts, 7.4 Integrating rational functions, 7.8 Improper Integrals

2. (Integral questions from class handouts)

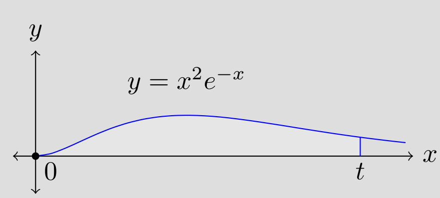
   (a) Without explicitly trying to evaluate this integral, determine whether it is possible for $\int_0^\infty x^2e^{-x}\,dx$ to be convergent to a negative value.

   Solution: It is not possible because $x^2e^{-x} \geq 0$ for all $x \geq 0$. See sketch in the solution of the next part.

   (b) Is $\int_0^\infty x^2e^{-x}\,dx$ convergent or divergent? If convergent, evaluate it.

   Solution: *Thinking about the problem:*

   The integral is $\lim_{t\to\infty} \int_0^t x^2e^{-x}\,dx$ and the graph of $y = x^2e^{-x}$ is below. We will compute $\int_0^t x^2e^{-x}\,dx$ and see how it behaves as $t \to \infty$.



To evaluate $\int_0^t x^2e^{-x}\,dx$ we will use integration by parts.

Doing the problem:

To work out $\int_0^t x^2e^{-x}\,dx$ with integration by parts set u and dv to be as in the chart below, and then compute du and v.

$u = x^2$	$dv = e^{-x}\,dx$
$du = 2x\,dx$	$v = -e^{-x}$

Thus $\int_0^t x^2e^{-x}\,dx = uv\Big|_0^t - \int_0^t v\,du = -x^2e^{-x}\Big|_0^t + \int_0^t 2xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx$. We work out the new integral also using integration by parts, starting with the chart below.

$u = x$	$dv = e^{-x}\,dx$
$du = dx$	$v = -e^{-x}$

Thus

$$\int_0^t xe^{-x}\,dx = -xe^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx = -\frac{t}{e^t} - e^{-x}\Big|_0^t = -\frac{t}{e^t} - \frac{1}{e^t} + 1,$$

so returning to the initial calculation we have

$$\int_0^t x^2e^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\left(-\frac{t}{e^t} - \frac{1}{e^t} + 1\right) = -\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2.$$

Letting $t \to \infty$, $\int_0^\infty x^2e^{-x}\,dx = \lim_{t\to\infty}\left(-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2\right) = 0 - 0 - 0 + 2$ by L'Hospital's rule (used twice for the first expression). Thus $\int_0^\infty x^2e^{-x}\,dx = 2$: the improper integral is convergent and equals 2.

(c) 1.) Evaluate $\int \ln\left(x + \sqrt{1+x^2}\right)$ dx or 2.) $\int x\ \tan^2 x$ dx or 3.) $\int \cos(\sqrt{x})$ dx or
4.) Evaluate $\int x^2\,(\ln x)^2$ dx or 5.) omitted or 6.) $\int \cos(\ln x)$ dx or
7a.) How can you derive the formula for Integration by Parts? or
7b.) Evaluate $\int_0^{\frac{\pi}{2}} x\ \cos(2x)$ dx or
7c.) Suppose $f(1)$=2, $f(4)$=7, $f'(1) = 5$, $f'(4)$=3. Suppose f'' is continuous. Evaluate $\int_1^4 x\ f''(x)$ dx or 7d.) Evaluate $\int \arctan x$ dx or 7e.) $\int e^x \cos x$ dx or

Solution: `https://egunawan.github.io/fall18/notes/hw7_1key.pdf`

(d) Evaluate 1.) $\int_0^\infty e^{-2x}$ dx or 2.) $\int_1^\infty \frac{1}{\sqrt{x}}$ or 3.) $\int_1^\infty \sin^2 x$ dx or 4.) $\int_1^\infty \frac{1}{x^2+2x-3}$ dx.

Solution: `https://egunawan.github.io/fall18/notes/LA7_8part1key.pdf`

(e) 1.) Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}}$ dx or 2.) Evaluate $\int_1^{\frac{\pi}{2}} \sec x$ or 3.) Evaluate $\int_0^5 \frac{x}{x-2}$ dx or
4.) Use the Comparison Theorem to determine whether $\int_1^\infty \frac{x}{x^3+1}$ dx converges or diverges.

Solution: `https://egunawan.github.io/fall18/notes/LA7_8part2key.pdf`

3. (Integrating rational functions)

(a) Evaluate $\displaystyle\int \frac{2x+1}{x^2-4}\,dx$.

Solution: Note: It is also possible to evaluate this using the trig substitution method, but the following will walk you through the Partial Fraction Decomposition method.

Thinking about the problem:

The integrand $\frac{2x+1}{x^2-4}$ is a rational function and does not look like it can be handled with substitution, so we use partial fractions. The denominator x^2-4 is $(x+2)(x-2)$, a product of different linear factors, so the partial fraction decomposition of $\frac{2x+1}{x^2-4}$ is $\frac{A}{x+2}+\frac{B}{x-2}$ for some constants A and B. After solving for A and B we would have $\int \frac{2x+1}{x^2-4}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x-2}\,dx$ and can integrate the right side.

Doing the Problem:

Writing $\frac{2x+1}{x^2-4} = \frac{A}{x+2}+\frac{B}{x-2}$, solve for A and B by multiplying both sides by the denominator x^2-4:

$$2x+1 = A(x-2)+B(x+2).$$

Setting $x=2$, we find

$$2(2)+1 = 5 = A(0)+B(2+2) = 4B \Rightarrow 5 = 4B \Rightarrow B = \frac{5}{4}.$$

Setting $x=-2$, we find

$$2(-2)+1 = A(-2-2)+B(0) \Rightarrow -3 = -4A \Rightarrow A = \frac{3}{4}.$$

Therefore $\frac{2x+1}{x^2-4} = \frac{3/4}{x+2}+\frac{5/4}{x-2}$, so

$$\begin{aligned}\int \frac{2x+1}{x^2-4}\,dx &= \frac{3}{4}\int \frac{dx}{x+2} + \frac{5}{4}\int \frac{dx}{x-2}\\ &= \boxed{\frac{3}{4}\ln|x+2| + \frac{5}{4}\ln|x-2| + C}.\end{aligned}$$

(b) Evaluate $\displaystyle\int \frac{9}{(x-6)(x+3)}\,\mathrm{dx}$ or $\displaystyle\int \frac{12}{(x-2)(x+1)}\,\mathrm{dx}$ or $\displaystyle\int \frac{8}{(x-1)(x+3)}\,\mathrm{dx}$.

Solution: Answer: After applying partial fraction decomposition, the integrand is equal to $\frac{1}{x-6}-\frac{1}{x+3}$ or $\frac{4}{x-2}-\frac{4}{x+1}$ or $\frac{2}{x-1}-\frac{2}{x+3}$.

The antiderivative is $\boxed{\ln\left|\frac{x-6}{x+3}\right|+C}$ or $\boxed{4\ln\left|\frac{x-2}{x+1}\right|+C}$ or $\boxed{2\ln\left|\frac{x-1}{x+3}\right|+C}$.

(c) Determine whether $\displaystyle\int_2^\infty \frac{1}{x^2+8x-9}\,\mathrm{dx}$ is convergent or divergent. If it is convergent, evaluate it.

Solution: Use partial fraction decomposition (probably faster) or complete the square + trig substitution (probably longer). Answer: $\boxed{\ln(11)/10}$.

3 11.3 Integral Test and Estimates of Sum

4. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

   a. Explain why the integral test can be applied.

   Solution: Let $f(x) = \frac{1}{x(\ln x)^5}$. Then $f(x)$ is continuous and positive on for $x \geq 2$. It is also decreasing on $[2, \infty)$ since $x(\ln x)^5$ is a product of increasing functions on $[2, \infty)$. Thus we can use the integral test.

   b. Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.

   Solution: Use u-substitution $u = \ln(x)$.

   c. Evaluate $\int_2^{\infty} \frac{1}{x(\ln x)^5}$.

   Solution: Use u-substitution $u = \ln(x)$.

   d. Apply the integral test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

   Solution: Since $\int_2^{\infty} \frac{dx}{x(\ln x)^5}$ converges by part (c), we conclude by the integral test that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ also converges.

   e. (Integral test and estimates of sums) Consider the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

   a. Verify that the integral test *can* be used to decide if this series converges.

   Solution:

   Let $f(x) = x/3^x$. We will show that $f(x)$ is positive, continuous, and decreasing on $[1, \infty)$.

   This function is positive and continuous for $x \geq 1$ since x is a polynomial, 3^x is an exponential function, and 3^x is never 0 on $[1, \infty)$.

   To show that $f(x)$ is decreasing on $[1, \infty)$, compute $f'(x) = \frac{1 - x\ln 3}{3^x}$, which implies $f'(x) < 0$ for $x > \frac{1}{\ln 3}$. Since $\ln 3 > \ln e = 1$, we have $\frac{1}{\ln 3} < 1$, so $f'(x) < 0$ for $x \leq 1$. This justifies the use of the integral test on $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

   b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.

Solution: compute $\int \frac{x}{3^x}\,dx$, try integration by parts:
either $\boxed{u = x \text{ and } dv = dx/3^x = 3^{-x}\,dx}$ or $\boxed{u = dx/3^x = 3^{-x} \text{ and } dv = xdx}$.
You should get

$$\int_1^\infty \frac{x}{3^x}\,dx = \lim_{b\to\infty}\left(-\frac{b}{3^b\ln 3} - \frac{1}{3^b(\ln 3)^2} + \frac{1}{3\ln 3} + \frac{1}{3(\ln 3)^2}\right).$$

Since $\lim_{b\to\infty} b/3^b = 0$ by L'Hospital's Rule and $\lim_{n\to\infty} 1/3^b = 0$,

$$\int_1^\infty \frac{x}{3^x}\,dx = \frac{1}{3\ln 3} + \frac{1}{3(\ln 3)^2}.$$

Since $\int_1^\infty \frac{x}{3^x}\,dx$ converges, the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$ also converges by the Integral Test.

c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

Solution: The Nth remainder R_N is at most $\int_N^\infty \frac{x}{3^x}\,dx = \lim_{b\to\infty}\int_N^b \frac{x}{3^x}\,dx = \frac{N\ln 3 + 1}{3^N(\ln 3)^2}$.
See Week 8 day 1 notes
`https://egunawan.github.io/fall18/lecture_notes_f18/f18_week8_day1_hw11_3.pdf`

d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

Solution: See Week 8 day 1 notes
`https://egunawan.github.io/fall18/lecture_notes_f18/f18_week8_day1_hw11_3.pdf`

5. Consider the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

a. Verify that the integral test *can* be used to decide if this series converges.

Solution:
Let $f(x) = x/3^x$. We will show that $f(x)$ is positive, continuous, and decreasing on $[1,\infty)$.

This function is positive and continuous for $x \geq 1$ since x is a polynomial, 3^x is an exponential function, and 3^x is never 0 on $[1,\infty)$.

To show that $f(x)$ is decreasing on $[1,\infty)$, compute $f'(x) = \frac{1 - x\ln 3}{3^x}$, which implies $f'(x) < 0$ for $x > \frac{1}{\ln 3}$. Since $\ln 3 > \ln e = 1$, we have $\frac{1}{\ln 3} < 1$, so $f'(x) < 0$ for $x \leq 1$. This justifies the use of the integral test on $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.

Solution: compute $\int \frac{x}{3^x}\,dx$, try integration by parts:
either $\boxed{u = x \text{ and } dv = dx/3^x = 3^{-x}\,dx}$ or $\boxed{u = dx/3^x = 3^{-x} \text{ and } dv = xdx}$.
You should get

$$\int_1^{\infty} \frac{x}{3^x}\,dx = \lim_{b\to\infty}\left(-\frac{b}{3^b \ln 3} - \frac{1}{3^b(\ln 3)^2} + \frac{1}{3\ln 3} + \frac{1}{3(\ln 3)^2}\right).$$

Since $\lim_{b\to\infty} b/3^b = 0$ by L'Hospital's Rule and $\lim_{n\to\infty} 1/3^b = 0$,

$$\int_1^{\infty} \frac{x}{3^x}\,dx = \frac{1}{3\ln 3} + \frac{1}{3(\ln 3)^2}.$$

Since $\int_1^{\infty} \frac{x}{3^x}\,dx$ converges, the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$ also converges by the Integral Test.

c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

Solution: The Nth remainder R_N is at most $\int_N^{\infty} \frac{x}{3^x}\,dx = \lim_{b\to\infty} \int_N^b \frac{x}{3^x}\,dx = \frac{N\ln 3 + 1}{3^N(\ln 3)^2}$.
See Week 8 day 1 notes
`https://egunawan.github.io/fall18/lecture_notes_f18/f18_week8_day1_hw11_3.pdf`

d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

Solution: See Week 8 day 1 notes
`https://egunawan.github.io/fall18/lecture_notes_f18/f18_week8_day1_hw11_3.pdf`

6. (Integral Test from 11.3 WebAssign)

(a) Evaluate the integral $\int_1^{\infty} \frac{3}{x^6}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^{\infty} \frac{3}{n^6}$ is convergent or divergent.

Solution: Answer: $= \boxed{3/5}$

(b) Evaluate the integral $\int_1^{\infty} \frac{1}{(4x+2)^3}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^{\infty} \frac{1}{(4n+2)^3}$ is convergent or divergent.

Solution: Answer: $= \boxed{1/288}$

(c) Evaluate the integral

$$\int_1^{\infty} x\ e^{-9x}\ \mathrm{dx}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^{\infty} \frac{n}{e^{9n}}$ is convergent or divergent.

Solution: Answer: $\boxed{10/81e^9}$

7. (Section 11.3 True/False)

(a) Is the following statement true or false? Justify.
Suppose $f(x)$ is a continuous function defined on $[5, \infty)$. If $f(x)$ is not bounded on $[5, \infty)$, we cannot apply the integral test using $\int_5^{\infty} f(x)\ \mathrm{dx}$. .

Solution: Answer to 7(a) is True. Justification: If $g(x)$ is positive and decreasing on $[5, \infty)$, then it is bounded (for example, bounded below by 0 and bounded above by $g(5)$). The contrapositive of this statement is: If $g(x)$ is not bounded, then it is not positive or not decreasing. Since $g(x)$ does not meet at least one of the criteria for applying the integral test using $\int_5^{\infty} g(x)\ \mathrm{dx}$, we cannot apply the integral test using $\int_5^{\infty} g(x)\ \mathrm{dx}$.

4 11.5 Alternating Series and Alt. Ser. Estimation Thm

8. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. Recall that the symbol 0! means the number 1.

(a) Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges or diverges.

Solution: Answer: Sec 11.5, Example 4 on page 735.

(b) Let $b_n = \frac{1}{n!}$. Your computing tool has computed for you $b_7 = \frac{1}{5040}$. What N do you need to use so that the partial sum S_N is correct (to the actual sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$) to three decimal places? Translation: we want $|S_N - S| < 0.0005$.

Solution: Answer: $\boxed{\text{N=6}}$. Alternating Estimation Theorem tells you that $S - S_6 \le b_7$, and I've computed for you $b_7 = \frac{1}{5040}$. Since

$$b_7 = \frac{1}{5040} < \frac{2}{10000} = 0.0002 < 0.0005,$$

we know that we only need to compute the partial sum $S_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!}$. More detailed explanation in Sec 11.5, Example 4 on page 735.

9. For the following questions, circle TRUE or FALSE. Justify briefly.

(a) Suppose $b_k > 0$ for all k and $\sum_{k=1}^{\infty}(-1)^k\, b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \leq b_6$. **T** **F**

Solution: Answer: True, by the Alternating Series Estimate Theorem

(b) Suppose $b_k > 0$ for all n and $\sum_{k=1}^{\infty}(-1)^k\, b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \geq b_6$. **T** **F**

Solution: Answer: False. The Alternating Series Estimate Theorem states that the inequality should go the other way.

10. Consider the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. Circle all true statement/s and cross out all false statement/s. (Hint: See the theorems on the exam's fact sheet).

a. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges.

Solution: $\boxed{\text{True}}$ by the Alternating Series Test.
Thinking about the problem:
The series starts off as $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ and is alternating with $b_n = \frac{1}{2n-1}$. We will check the conditions for the Alternating Series Test.
Doing the problem:
For $b_n = \frac{1}{2n-1} > 0$ we need to check $b_{n+1} \leq b_n$ for all n and $b_n \to 0$ as $n \to \infty$.
The inequality $b_{n+1} \leq b_n$ is the same as $\frac{1}{2n+1} \leq \frac{1}{2n-1}$, which is equivalent to saying $2n+1 \geq 2n-1$, and that last inequality is true. Alternatively, using calculus, the function $f(x) = \frac{1}{2x-1}$ has derivative $f'(x) = -\frac{2}{(2x-1)^2}$, which is negative for $x \geq 1$, so $f(x)$ is decreasing for $x \geq 1$.
For the limit, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$.
We can now use the Alternating Series Test to conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges.

b. The series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^3}$ converges.

Solution: $\boxed{\text{True}}$ by the Alternating Series Test.
Thinking about the problem:
The series is alternating with $b_n = \dfrac{1}{n^3}$. We will check the conditions for the Alternating Series Test.
Doing the problem:
For $b_n = \dfrac{1}{n^3} > 0$ we need to check $b_{n+1} \leq b_n$ for all n and $b_n \to 0$ as $n \to \infty$.
The inequality $b_{n+1} \leq b_n$ is the same as $\dfrac{1}{(n+1)^3} \leq \dfrac{1}{n^3}$, which is equivalent to saying $n+1 \geq n$, and that last inequality is true. Alternatively, using calculus, the function $f(x) = \dfrac{1}{x^3}$ has derivative $f'(x) = -\dfrac{3}{x^4}$, which is negative for $x \geq 2$, so $f(x)$ is decreasing for $x \geq 2$.
For the limit, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1}{n^3} = 0$.
We can now use the Alternating Series Test to conclude that $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^3}$ converges.

c. Suppose $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ and $S_{1000} := \displaystyle\sum_{n=1}^{1000} \frac{(-1)^{n-1}}{2n-1}$, $S_{1001} := \displaystyle\sum_{n=1}^{1001} \frac{(-1)^{n-1}}{2n-1}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$

Solution: $\boxed{\text{True}}$. Explanation: Every alternating series whose terms in absolute value satisfy $b_{n+1} < b_n$ lies in between consecutive partial sums. See Sec 11.5, Figs. 1 and 2, pg 733-734. Thus S is between S_{1000} and S_{1001}. The first term of $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^{n-1}}{2n-1}$ is positive, so $S_{1000} < S < S_{1001}$.

d. Suppose $S = \displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ and $S_{1000} := \displaystyle\sum_{n=2}^{1000} (-1)^n \frac{1}{n^3}$, $S_{1001} := \displaystyle\sum_{n=2}^{1001} (-1)^n \frac{1}{n^3}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$

Solution: $\boxed{\text{True}}$. Explanation: Every alternating series whose terms in absolute value satisfy $b_{n+1} < b_n$ lies in between consecutive partial sums. See Sec 11.5, Figs. 1 and 2, pg 733-734. Thus S is between S_{1000} and S_{1001}. The first term of $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ is positive, so $S_{1000} < S < S_{1001}$.

5 11.8 power series

11. What is the radius of convergence of a power series? What are the different possibilities?

Solution: There are three cases. See Sec 11.8, top of page 749

12. From textbook: Find the radius of convergence and interval of convergence of the following series

$$\text{(a.)} \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}. \quad \text{(b.)} \sum_{n=0}^{\infty} n! x^{2n}. \quad \text{(c.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n^5}. \quad \text{(d.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}.$$

Solution: (a.) See 11.8 Ex. 5, pg 750. (b.) Sec 11.8 Ex. 1, pg 747. (c.) Same radius of convergence as 11.8 Ex. 2, pg 747, but both endpoints are included. (d.) Same answer as 11.8 Ex. 3, pg 748.

13. Find the radius R and interval I of convergence of each series.

$$\text{(A.)} \sum_{n=1}^{\infty} \frac{x^n}{6n-1}. \quad \text{(B.)} \sum_{n=1}^{\infty} \frac{6^n(x+7)^n}{\sqrt{n}}. \quad \text{(C.)} \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n+4}. \quad \text{(D.)} \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n}}{(2n)!}.$$

Solution: (A.) $R = 1$, $I = [-1, 1)$. (B.) $R = 1/6$, $I = [-43/6, -41/6)$. (C.) $R = 1$, $I = (-1, 1]$. (D.) $R = \infty$, $I = (-\infty, \infty)$

14. (a) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is 16. What is the radius of convergence of the power series $\sum c_n x^{4n}$?

Solution: Answer: $\boxed{\sqrt[4]{16} = 2}$

(b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{5n}$?

Solution: Answer: $\boxed{\sqrt[5]{R}}$

15. Determine the radius of convergence and interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$.

Solution:

Thinking about the problem: The series converges at $x = 0$. For $x \neq 0$, we will investigate convergence at x using the Ratio Test ($a_n = (2^n/n)x^n$). After the interval of convergence is determined, convergence at its endpoints will be checked by other methods.

Doing the problem:

The problem asks for an interval of convergence of a power series. This series is $\sum_{n=1}^{\infty} a_n$ where $a_n = (2^n/n)x^n = 2^n x^n/n$, so for $x \neq 0$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n/n}\right| = \lim_{n\to\infty} 2|x|\frac{n}{n+1} = 2|x|.$$

By the Ratio Test, the series converges when $2|x| < 1$ and diverges when $2|x| > 1$, so the series converges when $|x| < \frac{1}{2}$ and diverges when $|x| > \frac{1}{2}$. Therefore the radius of convergence is $\boxed{R = \frac{1}{2}}$. The inequality $2|x| < 1$ says x is in $(-1/2, 1/2)$, and we need to test the endpoints to see if the power series converges when $x = -1/2$ or $x = 1/2$. First we let $x = 1/2$. Then the power series is

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

which diverges (the harmonic series). Next we let $x = -1/2$, so the power series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test. Therefore the interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$ is $\left[-\frac{1}{2}, \frac{1}{2}\right)$: the left endpoint is included but the right endpoint is not.

6 11.9 power series: using geometric series and step-by-step integration/differentiation

16. For each function, find a power series representation and determine the *interval* of convergence. (You can check your work with WolframAlpha. Type "series representation of ...")

    (a) $f(x) = \dfrac{x^3}{5+x}$

    Solution: See Sec 11.9 Example 3.

17. For each function, find a power series representation. Determine the *radius* of convergence.

    (a) $f(x) = \ln(1+x)$

    Solution: (see Sec 11.9, Example 6)

    (b) $\displaystyle\int \frac{1}{1+x^7}\,\mathrm{dx}$

Solution: (see Sec 11.9, Example 8)

18. (a) If the interval of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\sum_{n=1}^{\infty} n\, c_n x^{n-1}$? Why?

Solution: Answer: $\boxed{R = 10}$ by Theorem 'term-by-term differentiation' Sec 11.9.

(b) If the interval of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Why?

Solution: Answer: $\boxed{R = 10}$ by Theorem 'term-by-term integration' Sec 11.9.

19. Find a power series centered at $x = 0$ for the function $\frac{1}{2-5x}$ and find its interval of convergence.

Solution: *Thinking about the problem:* The function $f(x) = \frac{1}{2-5x}$ looks similar to $\frac{1}{1-x}$, so we will alter $f(x)$ to make it more closely resemble that. Factor 2 from the whole denominator: $\frac{1}{2-5x} = \frac{1}{2} \cdot \frac{1}{1-5x/2}$. We will write $\frac{1}{1-5x/2}$ as a geometric series by replacing x in $\frac{1}{1-x}$ with $5x/2$. The interval of convergence of the power series for $\frac{1}{1-x}$ is $(-1, 1)$, and we will use this to find the interval of convergence of the power series for $\frac{1}{1-5x/2}$, which will give us the interval of convergence for the power series of $f(x)$ centered at $x = 0$.

Doing the problem:

The problem is to find a power series of $f(x) = \frac{1}{2-5x}$ centered at $x = 0$. Write

$$f(x) = \frac{1}{2-5x} = \frac{1}{2(1-5x/2)} = \frac{1}{2} \cdot \frac{1}{1-5x/2}.$$

In the power series representation $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, replace x with $\frac{5x}{2}$:

$$\frac{1}{1-5x/2} = \sum_{n=0}^{\infty} \left(\frac{5x}{2}\right)^n \quad \text{for} \quad \left|\frac{5x}{2}\right| < 1.$$

Thus

$$f(x) = \frac{1}{2-5x} = \frac{1}{2} \cdot \frac{1}{1-5x/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{5x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty}\frac{5^n}{2^{n+1}}x^n \text{ for } \left|\frac{5x}{2}\right| < 1}.$$

We have found a power series for $f(x)$ centered at $x = 0$ and that it converges precisely when $\left|\frac{5x}{2}\right| < 1$, which is the same as $|x| < \frac{2}{5}$. Therefore a power series for $f(x)$ centered at $x = 0$ has interval of convergence $\boxed{\left(-\frac{2}{5}, \frac{2}{5}\right)}$.