1 4.4 Limit laws and L'Hospital's Rule

1. i.) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   ii.) If $\lim_{x\to\infty} f(x) = 1$, then $\lim_{x\to\infty} (f(x))^x$ is ...
   (a) zero (b) ∞ (c) 1 (d) another method is needed to determine this.

   iii.) Evaluate $\lim_{x\to 1} \dfrac{\ln x}{x-1}$ and $\lim_{x\to\infty} \dfrac{e^x}{x^2}$.

   iv.) Evaluate $\lim_{x\to 0^+} x(\ln x)^3$. Note: This shows up frequently when we compute our improper integral examples.

2 7.1 Integration by Parts, 7.4 Integrating rational functions, 7.8 Improper Integrals

2. (Integral questions from class handouts)

   (a) Without explicitly trying to evaluate this integral, determine whether it is possible for $\int_0^\infty x^2 e^{-x}\,dx$ to be convergent to a negative value.

   (b) Is $\int_0^\infty x^2 e^{-x}\,dx$ convergent or divergent? If convergent, evaluate it.

   (c) 1.) Evaluate $\int \ln\left(x + \sqrt{1+x^2}\right)\,\mathrm{dx}$ or 2.) $\int x\ \tan^2 x\ \mathrm{dx}$ or 3.) $\int \cos(\sqrt{x})\ \mathrm{dx}$ or

   4.) Evaluate $\int x^2 (\ln x)^2\ \mathrm{dx}$ or 5.) omitted or 6.) $\int \cos(\ln x)\ \mathrm{dx}$ or

   7a.) How can you derive the formula for Integration by Parts? or

   7b.) Evaluate $\int_0^{\frac{\pi}{2}} x\ \cos(2x)\ \mathrm{dx}$ or

   7c.) Suppose $f(1)$=2, $f(4)$=7, $f'(1) = 5$, $f'(4)$=3. Suppose f'' is continuous. Evaluate $\int_1^4 x\ f''(x)\ \mathrm{dx}$

   or 7d.) Evaluate $\int \arctan x\ \mathrm{dx}$ or 7e.) $\int e^x \cos x\ \mathrm{dx}$ or

   (d) Evaluate 1.) $\int_0^\infty e^{-2x}\ \mathrm{dx}$ or 2.) $\int_1^\infty \dfrac{1}{\sqrt{x}}$ or 3.) $\int_1^\infty \sin^2 x\ \mathrm{dx}$ or 4.) $\int_1^\infty \dfrac{1}{x^2+2x-3}\ \mathrm{dx}$.

   (e) 1.) Evaluate $\int_0^8 \dfrac{1}{\sqrt[3]{x}}\ \mathrm{dx}$ or 2.) Evaluate $\int_1^{\frac{\pi}{2}} \sec x$ or 3.) Evaluate $\int_0^5 \dfrac{x}{x-2}\ \mathrm{dx}$ or

   4.) Use the Comparison Theorem to determine whether $\int_1^\infty \dfrac{x}{x^3+1}\ \mathrm{dx}$ converges or diverges.

3. (Integrating rational functions)

   (a) Evaluate $\int \dfrac{2x+1}{x^2-4}\,dx$.

   (b) Evaluate $\int \dfrac{9}{(x-6)(x+3)}\ \mathrm{dx}$ or $\int \dfrac{12}{(x-2)(x+1)}\ \mathrm{dx}$ or $\int \dfrac{8}{(x-1)(x+3)}\ \mathrm{dx}$.

   (c) Determine whether $\int_2^\infty \dfrac{1}{x^2+8x-9}\ \mathrm{dx}$ is convergent or divergent. If it is convergent, evaluate it.

3 11.3 Integral Test and Estimates of Sum

4. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

   a. Explain why the integral test can be applied.

   b. Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.

   c. Evaluate $\int_2^{\infty} \frac{1}{x(\ln x)^5}$.

   d. Apply the integral test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

5. Consider the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

   a. Verify that the integral test *can* be used to decide if this series converges.

   b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.

   c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

   d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

6. (Integral Test from 11.3 WebAssign)

   (a) Evaluate the integral $\int_1^{\infty} \frac{3}{x^6}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^{\infty} \frac{3}{n^6}$ is convergent or divergent.

   (b) Evaluate the integral $\int_1^{\infty} \frac{1}{(4x+2)^3}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^{\infty} \frac{1}{(4n+2)^3}$ is convergent or divergent.

   (c) Evaluate the integral

   $$\int_1^{\infty} x\, e^{-9x}\ \mathrm{dx}$$

   Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^{\infty} \frac{n}{e^{9n}}$ is convergent or divergent.

7. (Section 11.3 True/False)

   (a) Is the following statement true or false? Justify.
   Suppose $f(x)$ is a continuous function defined on $[5, \infty)$. If $f(x)$ is not bounded on $[5, \infty)$, we cannot apply the integral test using $\int_5^{\infty} f(x)$ dx. .

4 11.5 Alternating Series and Alt. Ser. Estimation Thm

8. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. Recall that the symbol 0! means the number 1.

   (a) Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges or diverges.

   (b) Let $b_n = \frac{1}{n!}$. Your computing tool has computed for you $b_7 = \frac{1}{5040}$. What N do you need to use so that the partial sum S_N is correct (to the actual sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$) to three decimal places? Translation: we want $|S_N - S| < 0.0005$.

9. For the following questions, circle TRUE or FALSE. Justify briefly.

   (a) Suppose $b_k > 0$ for all k and $\sum_{k=1}^{\infty}(-1)^k\ b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \leq b_6$. **T** **F**

   (b) Suppose $b_k > 0$ for all n and $\sum_{k=1}^{\infty}(-1)^k\ b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \geq b_6$. **T** **F**

10. Consider the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. Circle all true statement/s and cross out all false statement/s. (Hint: See the theorems on the exam's fact sheet).

    a. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges.

    b. The series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^3}$ converges.

    c. Suppose $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ and $S_{1000} := \sum_{n=1}^{1000} \frac{(-1)^{n-1}}{2n-1}$, $S_{1001} := \sum_{n=1}^{1001} \frac{(-1)^{n-1}}{2n-1}$ are partial sums, as usual. Then is the following True or False, and why?

    $$S_{1000} < S < S_{1001}.$$

    d. Suppose $S = \sum_{n=2}^{\infty}(-1)^n \frac{1}{n^3}$ and $S_{1000} := \sum_{n=2}^{1000}(-1)^n \frac{1}{n^3}$, $S_{1001} := \sum_{n=2}^{1001}(-1)^n \frac{1}{n^3}$ are partial sums, as usual. Then is the following True or False, and why?

    $$S_{1000} < S < S_{1001}.$$

5 11.8 power series

11. What is the radius of convergence of a power series? What are the different possibilities?

12. From textbook: Find the radius of convergence and interval of convergence of the following series

$$\text{(a.)} \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}. \quad \text{(b.)} \sum_{n=0}^{\infty} n! x^{2n}. \quad \text{(c.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n^5}. \quad \text{(d.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}.$$

13. Find the radius R and interval I of convergence of each series.

$$\text{(A.)} \sum_{n=1}^{\infty} \frac{x^n}{6n-1}. \quad \text{(B.)} \sum_{n=1}^{\infty} \frac{6^n (x+7)^n}{\sqrt{n}}. \quad \text{(C.)} \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n+4}. \quad \text{(D.)} \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n}}{(2n)!}.$$

14. (a) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is 16. What is the radius of convergence of the power series $\sum c_n x^{4n}$?

    (b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{5n}$?

15. Determine the radius of convergence and interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$.

6 11.9 power series: using geometric series and step-by-step integration/differentiation

16. For each function, find a power series representation and determine the *interval* of convergence. (You can check your work with WolframAlpha. Type “series representation of ...”)

    (a) $f(x) = \dfrac{x^3}{5+x}$

17. For each function, find a power series representation. Determine the *radius* of convergence.

    (a) $f(x) = \ln(1+x)$

    (b) $\displaystyle\int \frac{1}{1+x^7}\,\mathrm{dx}$

18. (a) If the interval of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} n\, c_n x^{n-1}$? Why?

    (b) If the interval of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Why?

19. Find a power series centered at $x = 0$ for the function $\dfrac{1}{2-5x}$ and find its interval of convergence.