Contents

1 4.4 Limit laws and L'Hospital's Rule

1. i.) If $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} (xf(x))$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   ii.) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   iii.) If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   iv.) If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0} (f(x) + g(x))$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   v.) If $\lim_{x\to\infty} f(x) = 1$, then $\lim_{x\to\infty} (f(x))^x$ is ...
   (a) zero (b) ∞ (c) 1 (d) another method is needed to determine this.

   vi.) Evaluate $\lim_{x\to 1} \frac{\ln x}{x-1}$ and $\lim_{x\to\infty} \frac{e^x}{x^2}$.

   vii.) Evaluate $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ and $\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

   viii.) Evaluate $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$.

   ix.) Evaluate $\lim_{x\to 0^+} x\ln x$. Note: This shows up frequently when we compute our improper integral examples.

   x.) Evaluate $\lim_{x\to 0^+} x(\ln x)^3$. Note: This shows up frequently when we compute our improper integral examples.

2 7.1 Integration by Parts, 7.4 Integrating rational functions, 7.8 Improper Integrals

2. (Integral questions from class handouts)

   (a) Consider $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$, $\int_{-\sqrt{2}}^{0} \frac{x^2}{\sqrt{4-x^2}}dx$, $\int_0^{2} \frac{x^2}{\sqrt{4-x^2}}dx$ and $\int_{-2}^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$. Which are proper integrals and which are improper integrals?

(b) Without explicitly trying to evaluate this integral, determine whether it is possible for $\int_0^\infty x^2 e^{-x}\,dx$ to be convergent to a negative value.

(c) Is $\int_0^\infty x^2 e^{-x}\,dx$ convergent or divergent? If convergent, evaluate it.

(d) 1.) Evaluate $\int \ln\left(x+\sqrt{1+x^2}\right)$ dx or 2.) $\int x\ \tan^2 x$ dx or 3.) $\int \cos(\sqrt{x})$ dx or

4.) Evaluate $\int x^2 (\ln x)^2$ dx or 5.) omitted or 6.) $\int \cos(\ln x)$ dx or

7a.) How can you derive the formula for Integration by Parts? or

7b.) Evaluate $\int_0^{\frac{\pi}{2}} x\ \cos(2x)$ dx or

7c.) Suppose $f(1)$=2, $f(4)$=7, $f'(1)=5$, $f'(4)$=3. Suppose f'' is continuous. Evaluate $\int_1^4 x\ f''(x)$ dx

or 7d.) Evaluate $\int \arctan x$ dx or 7e.) $\int e^x \cos x$ dx or

7f.) A particle that moves along a straight line has velocity $v(t)=t^3e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

(e) Evaluate 1.) $\int_0^\infty e^{-2x}$ dx or 2.) $\int_1^\infty \frac{1}{\sqrt{x}}$ or 3.) $\int_1^\infty \sin^2 x$ dx or 4.) $\int_1^\infty \frac{1}{x^2+2x-3}$ dx.

(f) 1.) Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}}$ dx or 2.) Evaluate $\int_1^{\frac{\pi}{2}} \sec x$ or 3.) Evaluate $\int_0^5 \frac{x}{x-2}$ dx or

4.) Use the Comparison Theorem to determine whether $\int_1^\infty \frac{x}{x^3+1}$ dx converges or diverges.

3. (Various definite and indefinite integrals).

(a) Evaluate $\int e^{\sqrt{x}}$ dx and $\int_0^1 e^{\sqrt{x}}$ dx and $\int_1^\infty e^{\sqrt{x}}$ dx

(b) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(c) Is $\int_1^2 t^3 \ln(t)dt$ a proper or improper integral?

(d) Evaluate $\int t^3 \ln(t)dt$ and $\int_1^2 t^3 \ln(t)dt$

(e) Perform either a proof or a reality check for the previous problem.

(f) Evaluate $\int \frac{\sin(\ln(x))}{x}$ dx and $\int_1^\infty \frac{\sin(\ln(x))}{x}$ dx

(g) Evaluate $\int (x+2)\sin(3x)$ dx

(h) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(i) Evaluate $\int \frac{x}{\sqrt{4-x^2}}$ and $\int_0^2 \frac{x}{\sqrt{4-x^2}}$

(j) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(k) Is $\int_0^9 \frac{1}{\sqrt{x}}$ dx an improper integral?

(l) Determine whether $\int_0^{25} \frac{1}{\sqrt{x}}$ dx or $\int_0^{16} \frac{1}{\sqrt{x}}$ dx or $\int_0^{9} \frac{1}{\sqrt{x}}$ dx is convergent or divergent. If it is convergent, evaluate the integral.

4. (Integrating rational functions)

(a) Evaluate $\int \frac{2x+1}{x^2-4}\,dx$.

(b) Evaluate $\int \frac{10}{(x+5)(x-2)}$ dx.

(c) Provide a computation that is either a formal verification (that is, a proof) or simply a sanity-check for your answer to the previous question.

(d) Evaluate $\int \frac{9}{(x-6)(x+3)}$ dx or $\int \frac{12}{(x-2)(x+1)}$ dx or $\int \frac{8}{(x-1)(x+3)}$ dx.

Evaluate $\int_7^{\infty} \frac{9}{(x-6)(x+3)}$ dx or $\int_7^{\infty} \frac{12}{(x-2)(x+1)}$ dx or $\int_7^{\infty} \frac{8}{(x-1)(x+3)}$ dx.

5. (Improper integrals Sec 7.8)

(a) When is an integral improper? Hint: There are two kinds. Copy definitions from Sec 7.8 pg 527 and 531.

(b) When is an integral proper? (The answer is when it is not improper, but explain what needs to happen for a definite integral to be proper).

(c) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^{\infty} f(x)$ dx $= \lim_{t\to\infty} \int_a^t f(x)$ dx **T** **F**

Justification:

(d) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^{\infty} f(x)$ dx $= \lim_{a\to\infty} \int_a^t f(x)$ dx **T** **F**

Justification:

(e) True or False, and why? The integral $\int_2^3 \sqrt{x-2}$ dx is improper. **T** **F**

(f) True or False, and why? $\int_0^1 \frac{27}{x^5}$ dx is improper. **T** **F**

(g) Evaluate $\int_0^1 \frac{27}{x^5}$ dx

(h) True or False? $\int_{-1}^1 \frac{1}{x}$ dx is improper. **T** **F**

(i) Evaluate $\int_{-1}^1 \frac{1}{x}$ dx

(j) Determine whether $\int_0^1 9x^2 \ln(x)$ dx converges or diverges. If it converges, evaluate it.

(k) Determine whether $\int_e^{\infty} \frac{1}{x(\ln x)^3}$ dx converges or diverges. If it converges, evaluate the integral.

(l) Determine whether
$$\int_2^{\infty} \left(\frac{1}{e^5}\right)^x \text{ dx}$$
is convergent or divergent. If it is convergent, evaluate it.

(m) Determine whether $\displaystyle\int_2^{\infty} \frac{1}{x^2+8x-9}$ dx is convergent or divergent. If it is convergent, evaluate it.

(n) Determine whether $\displaystyle\int_0^{1} \frac{4}{x^5}$ dx is convergent or divergent. If it is convergent, evaluate it.

(o) Determine whether $\displaystyle\int_0^{1} \frac{4}{x^{0.5}}$ dx is convergent or divergent. If it is convergent, evaluate it.

(p) Determine whether $\displaystyle\int_2^{3} \frac{2}{\sqrt{3-x}}$ dx is convergent or divergent. If it is convergent, evaluate it.

(q) Write 2 improper integrals (different from above) so that one is convergent and the other is divergent.

(r) Write 2 proper (definite) integrals that are different from above.

(s) Write 2 indefinite integrals.

6. Write down the first key step/s of evaluating the following the integrals (There often are more than one right answer). You don't need to evaluate the integrals.

(a) $\displaystyle\int_0^{4} \frac{\ln(x)}{\sqrt{x}}$ dx ______________________

(b) $\displaystyle\int_1^{2} \ln(x)$ dx ______________________

(c) $\int xe^{0.2x}$ dx ______________________

(d) $\displaystyle\int_0^{1} e^x \sin(x)$ dx ______________________

3 11.3 Integral Test and Estimates of Sum

7. Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

a. Explain why the integral test can be applied.

b. Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.

c. Evaluate $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^5}$.

d. Apply the integral test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

8. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$.

a. Verify that the integral test *can* be used to decide if this series converges.

b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.

c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

9. (Integral Test from 11.3 WebAssign)

(a) Find the values of p for which the integral $\int_e^\infty \frac{6}{x(\ln x)^p}$ dx converges. Evaluate the integral for these values of p.
(Hint: Your work should look like Example 4 on pg 530. Check the three cases, when $p = 1$, when $p < 1$, and when $p > 1$.)

(b) Evaluate the integral $\int_1^\infty \frac{3}{x^6}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{3}{n^6}$ is convergent or divergent.

(c) Evaluate the integral $\int_1^\infty \frac{1}{(4x+2)^3}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{1}{(4n+2)^3}$ is convergent or divergent.

(d) Evaluate the integral $\int_1^\infty \frac{1}{\sqrt{x+9}}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{1}{\sqrt{n+9}}$ is convergent or divergent.

(e) Evaluate the integral

$$\int_1^\infty x\ e^{-9x}\ \text{dx}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{n}{e^{9n}}$ is convergent or divergent.

(f) The following statement is false: "If $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^\infty a_n = \int_1^\infty f(x)\,dx$."
Give a counterexample by coming up with a continuous, positive, and decreasing $f(x)$ on $[1, \infty)$ and computing both $\sum_{n=1}^\infty a_n$ (where $a_n := f(n)$) and $\int_1^\infty f(x)\,dx$, showing that they are not equal.
(Hint: you know how to compute precisely the sum of any convergent geometric series).

10. (Section 11.3 True/False)

(a) Is the following statement true or false? Justify.
Suppose $f(x)$ is a continuous function defined on $[5, \infty)$. If $f(x)$ is not bounded on $[5, \infty)$, we cannot apply the integral test using $\int_5^\infty f(x)$ dx. .

11. (Sec 11.3 p-series and shifting indices)

(Note: the symbol ζ is the lower-case Greek letter which is pronounced "zeta" in English).
The *Riemann zeta*-function ζ is defined by

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

It is used in number theory to study the distribution of prime numbers.

(a) What is the domain of the function ζ? (That is, for what values of x is this function defined?) (Hint: go to Sec 11.3, page 722)

(b) Euler computed $\zeta(2)$ to be $\frac{\pi^2}{6}$. (See page 720, sec 11.3). Use this fact to find the sum of each series below. Hint: Given a convergent series, you can multiply out a constant, and subtract terms as needed.

$$\sum_{n=3}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(5n)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+3)^2}$$

4 11.5 Alternating Series and Alt. Ser. Estimation Thm

12. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. Recall that the symbol 0! means the number 1.

(a) Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges or diverges.

(b) Let $b_n = \frac{1}{n!}$. Your computing tool has computed for you $b_7 = \frac{1}{5040}$. What N do you need to use so that the partial sum S_N is correct (to the actual sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$) to three decimal places? Translation: we want $|S_N - S| < 0.0005$.

13. For the following questions, circle TRUE or FALSE. Justify briefly.

(a) Suppose $b_k > 0$ for all k and $\sum_{k=1}^{\infty} (-1)^k\, b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \le b_6$. **T** **F**

(b) Suppose $b_k > 0$ for all n and $\sum_{k=1}^{\infty} (-1)^k\, b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \ge b_6$. **T** **F**

(c) An application of the Alternating Ser. Estimation Thm is a proof that e is irrational. **T** **F**

(d) An application of the Alternating Ser. Estimation Thm is a proof that e is rational. **T** **F**

14. Consider the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. Circle all true statement/s and cross out all false statement/s. (Hint: See the theorems on the exam's fact sheet).

a. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges.

b. The series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ converges.

c. Suppose $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ and $S_{1000} := \sum_{n=1}^{1000} \frac{(-1)^{n-1}}{2n-1}$, $S_{1001} := \sum_{n=1}^{1001} \frac{(-1)^{n-1}}{2n-1}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$

d. Suppose $S = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ and $S_{1000} := \sum_{n=2}^{1000} (-1)^n \frac{1}{n^3}$, $S_{1001} := \sum_{n=2}^{1001} (-1)^n \frac{1}{n^3}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$

5 11.8 power series

15. What is a power series?

16. What is the radius of convergence of a power series? What are the different possibilities?

17. In most cases, how do you find the radius of convergence of a power series?

18. From textbook: Find the radius of convergence and interval of convergence of the following series

$$\text{(a.)} \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}. \quad \text{(b.)} \sum_{n=0}^{\infty} n!x^{2n}. \quad \text{(c.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n^5}. \quad \text{(d.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}.$$

19. Find the radius R and interval I of convergence of each series.

$$\text{(A.)} \sum_{n=1}^{\infty} \frac{x^n}{6n-1}. \quad \text{(B.)} \sum_{n=1}^{\infty} \frac{6^n(x+7)^n}{\sqrt{n}}. \quad \text{(C.)} \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n+4}. \quad \text{(D.)} \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n}}{(2n)!}.$$

20. (a) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is 16. What is the radius of convergence of the power series $\sum c_n x^{4n}$?

    (b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{5n}$?

21. Determine the radius of convergence and interval of convergence for $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$.

6 11.9 power series: using geometric series and step-by-step integration/differentiation

22. For each function, find a power series representation and determine the *interval* of convergence. (You can check your work with WolframAlpha. Type "series representation of ...")
    (a) $f(x) = \dfrac{1}{3+x}$
    (b) $f(x) = \dfrac{x^3}{5+x}$
    (c) $f(x) = \dfrac{x}{1+10x^2}$

23. For each function, find a power series representation. Determine the *radius* of convergence.
    (a) $f(x) = \dfrac{1}{(2+x)^2}$
    (b) $f(x) = \dfrac{1}{(2+x)^3}$
    (c) $f(x) = \dfrac{x}{(2+x)^3}$
    (d) $f(x) = \ln(1+x)$
    (e) $f(x) = arctan(x)$
    (f) $\displaystyle\int \frac{1}{1+x^7}\,\mathrm{dx}$
    (g) $\displaystyle\int \frac{x}{1-x^7}\,\mathrm{dx}$

24. (a) If the interval of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\sum_{n=1}^{\infty} n\, c_n x^{n-1}$? Why?
    (b) If the interval of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Why?

25. Find a power series centered at $x = 0$ for the function $\dfrac{1}{2-5x}$ and find its interval of convergence.