Math 1152, Fall 2018 — Exam 2 Fact Sheet

Useful trig facts. $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}, \quad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Some derivatives and antiderivatives.

$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dx}\cos(x) = -\sin(x)$	$\frac{d}{dx}\tan(x) = (\sec(x))^2$
$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$	$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$	$\frac{d}{dx}\cot(x) = -(\csc(x))^2$
$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$
$\frac{d}{dx}b^x = \ln(b)b^x$	$\int \sec x \, \mathrm{dx} = \ln\lvert\sec x + \tan x\rvert + C$	$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ if $a \neq 0$

L'Hôpital's rule. Suppose f, g are differentiable functions and $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are both 0 or both $\pm\infty$. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

Geometric series. A series of the form $\sum ar^n$ is called a **geometric series**.

- if $|r| < 1$ then the series converges and $\sum_{n=0}^{\infty} ar^n = a/(1-r)$.
- if $|r| \geq 1$ then the series diverges.

Divergence test. Consider the series $\sum a_n$. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ **diverges**.

p-series test. A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is called a p-series. The series converges if and only if $p > 1$.

Limit comparison test. Consider the series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and suppose that $\lim_{n\to\infty}\frac{a_n}{b_n} = c > 0$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Ratio test. Consider the series $\sum_{n=1}^{\infty} a_n$ and suppose that $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

1. if $L < 1$ then $\sum_{n=1}^{\infty}|a_n|$ converges.
2. if $L > 1$ or $L = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges.
3. if $L = 1$ then no conclusion can be drawn using the Ratio test.

Integration by parts fomula. $\int u\,dv = uv - \int v\,du$

Partial Fraction Decomposition.

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \text{ if } a \neq b, \quad \text{and} \quad \frac{1}{x(x^2+a)} = \frac{A}{x} + \frac{Bx+C}{x^2+a} \text{ if } a \neq 0$$

Integral test.
If f is continuous, non-negative, and decreasing on $[1, \infty)$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if the improper integral } \int_1^{\infty} f(x)\,dx \text{ converges.}$$

Remainder Estimate for the Integral Test.

If $a_n = f(n)$, where $f(x)$ is a continuous, positive, decreasing function for $x \geq 1$ as above, suppose $S = \sum_{n=1}^{\infty} a_n$ converges. For $N \geq 1$, let $s_N = \sum_{n=1}^{N} a_n$ and $R_N = S - S_N$, so R_N is the Nth remainder term. Then

$$\int_{N+1}^{\infty} f(x)\,dx \quad \leq \quad R_N \quad \leq \quad \int_N^{\infty} f(x)\,dx.$$

Alternating series test.
A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n b_n$$

where $b_n \geq 0$ for all n is called an alternating series. If

1. $b_{n+1} \leq b_n$ for all n large enough (ie. $\{b_n\}$ is an eventually decreasing sequence)
2. $\lim_{n\to\infty} b_n = 0$

then the series **converges**.

Alternating Series Estimation Theorem

If $S = \sum_{k=r}^{\infty} (-1)^k b_k$, where $b_k > 0$, is the sum of an alternating series that satisfies

$$\text{(i) } b_{k+1} \leq b_k \qquad \text{and} \qquad \text{(ii) } \lim_{k\to\infty} b_k = 0,$$

then $|R_N| = |S - S_N| \leq b_{N+1}$, where $S_N := \sum_{k=r}^{N} (-1)^k b_k$.