University of Connecticut
Department of Mathematics

MATH 1152Q — EXAM 1 PRACTICE — FALL 2018

- From each number, choose a couple parts that seem the most challenging.
- **You will earn a small amount of bonus 'style points' for a legible, coherent, and non-ambiguous paper. Your reader should not need to reread your solution several times to find a train of thought. In addition, you should use correct mathematical notations. This includes not writing an equal sign between two unequal objects, not treating the symbol ∞ like a number, and not attempting to multiply 0 with the symbol ∞.**
- Please read each question carefully. Show **ALL** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. No books or other references are permitted. All technology (phones, calculators) and books/ notes should be placed inside your bag.

1 Part A: Sequences and Series, Sec 11.1, 11.2, 11.4, 11.5, 11.6

1. For the following questions, circle TRUE or FALSE, and give a justification. True statements should be argued for using facts, theorems or definitions from class.

   (a) If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ converges. **T** **F**

   Justification:

   (b) If $a_n > 0$, $b_n > 0$ for all n, $\sum b_n$ diverges, and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ diverges. **T** **F**

   Justification:

   (c) If $a_n > 0$ for all n & $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, then $\sum a_n$ is convergent by the ratio test **T** **F**

   Justification:

   (d) If a_n and b_n are both positive for all n and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ is convergent by the limit comparison test **T** **F**

   Justification:

   (e) The harmonic series $\sum 1/n$ is convergent by the p-series test **T** **F**

   Justification:

   (f) We can use the ratio test *alone* to show the geometric series $\sum \frac{2^n}{3^n}$ converges **T** **F**

   Justification:

   (g) We can use the p-series test *alone* to show the series $\sum 2^n/3^n$ converges **T** **F**

   Justification:

   (h) We can apply the monotonic sequence theorem to show that the geometric **sequence** $\{2^n/3^n\}_{n=1}^{\infty}$ is convergent **T** **F**

   Justification:

   (i) We can apply the monotonic sequence theorem to show that the harmonic sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent **T** **F**

   Justification:

   (j) We can apply the squeeze theorem to show that the alternating harmonic sequence $\left\{\frac{(-1)^n}{n}\right\}$ is convergent **T** **F**

   Justification:

   (k) It is impossible for a subset of a line to have infinitely many points and have length zero. **T** **F**

   Justification:

(l) The divergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $0 < p < 1$ follows from divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the comparison test. **T** **F**
Justification:

(m) The convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$ follows from divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the comparison test. **T** **F**
Justification:

(n) Convergence of $\sum \frac{1}{n^p}$ for $p > 1$ can be shown with the ratio test. **T** **F**
Justification:

(o) Divergence of a p-series for $p < 1$ can be shown with the ratio test. **T** **F**
Justification:

2. (a) State the contrapositive of the factual statement: "If the sequence $\{a_n\}$ is unbounded, then it is divergent".

   (b) Is the contrapositive statement you wrote as your answer to part (a) true or false? **Justification (explain or give a counterexample):**

   (c) The converse of part (a) is the following: "If the sequence $\{a_n\}$ is divergent, then $\{a_n\}$ is unbounded". Is this true or false? **Justification (explain or give a counterexample):**

3. Answer the following on the line provided.

   (a) What is the 100th term of the sequence $\{2, 5, 8, 11, \ldots\}$?
   (The terms 2 and 5 are the first and second term, respectively) ____________

   (b) Find a formula for the general term a_n of the sequence $\left\{1, -\frac{2}{5}, \frac{3}{25}, -\frac{4}{125}, \frac{5}{625} \ldots\right\}$. Make sure to specify your starting value of n.

   (c) Write the geometric series $4 + 2 + 1 + \frac{1}{2} \cdots$ in standard form (using summation notation $\sum$). ____________

   (d) Find the 4th term a_4 in the recursive sequence $a_{n+1} = 2a_n + a_{n-1}$ when $a_1 = 1$ and $a_2 = 1$. ____________

   (e) Find the 7th term a_7 in the recursive sequence $a_{n+1} = a_n + a_{n-1}$ when $a_1 = 2$ and $a_2 = 3$. ____________

   (f) We can use geometric series to compute
   $$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots.$$
   What fraction is this equal to?

   (g) We can use geometric series to compute $0.9999\ldots$. What fraction is this equal to?

   (h) One of the two decimal expansions for a number is $2.449999\ldots$. What's the other?

   (i) Use geometric series to compute the fraction for $1.833333\ldots$.

   Perform a sanity check against your answer.

   (j) Use geometric series to compute the fraction for $1.08333333333333333\ldots$

   Perform a sanity check against your answer.

   (k) Does the series $\sum_{n=1}^{\infty} - \ln\left(\frac{n}{2n+7}\right)$ converge or diverge? ____________

   (l) Find the sum of the series $\sum_{k=0}^{\infty} 5\left(\frac{2}{3}\right)^k$. ____________

(m) Find the sum of the series $\sum_{n=2}^{\infty} 5\left(\frac{(-6)^{n-1}}{7^n}\right)$. ____________

Perform a sanity check against your answer.

(n) Find the sum of the series ____________

$$-5+3-\frac{9}{5}+\frac{27}{25}-\frac{81}{125}+\dots.$$

Perform a sanity check against your answer.

(o) Write an expression for the nth term in the sequence ____________

$\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots\right\}$. (The terms $\frac{1}{2}$ and $\frac{1}{6}$ are the first and second terms in the sequence)

(p) Write an equivalent series with index summation beginning at $n = 0$. ____________

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!}$$

Perform a sanity check against your answer.

(q) Write an equivalent series with index summation beginning at $n = 1$. ____________

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!}$$

Perform a sanity check against your answer.

(r) For what values of k does the series $\sum \frac{5}{n^k}$ converge? Please explain. ____________

(s) Find the values of A so that the series $\sum_{n=1}^{\infty} \frac{(A)^{n-1}}{3^{n-1}}$ is convergent. Please explain.

(t) Find the values of B so that the series $\sum_{n=1}^{\infty} \frac{(B-3)^{n-1}}{3^{n-1}}$ is convergent. Please explain.

(u) Find the values of C so that the series $\sum_{n=5}^{\infty} \frac{(C-2)^{n}}{3^{n+1}}$ is convergent. Please explain.

4. (a) (copy from Sec 11.1 page 696) Let $\{a_n\}$ be a sequence and let $L \in \mathbb{R}$ (this notation means that L is a real number). What does $\lim_{n\to\infty} a_n = L$ mean?

(b) (copy from Sec 11.1 page 696) Let $\{a_n\}$ be a sequence and let $L \in \mathbb{R}$ (this notation means that L is a real number). What does $\lim_{n\to\infty} a_n = L$ mean? Use the $\epsilon - N$ definition.

(c) (copy from Sec 11.2, page 708) Let $\{c_n\}$ be a sequence. What is a partial sum of $\{c_n\}$?

(d) (copy from Sec 11.2, page 708) Let $\{c_n\}_{n=1}^{\infty}$ be a sequence. We say that the infinite series $\sum_{n=1}^{\infty} c_n$ is *convergent* if

____________________________________. (Hint: your answer should include the words 'limit' and 'partial sums') If the above blank is not true, then we say that $\sum_{n=1}^{\infty} c_n$ is *not convergent* or *divergent*.

(e) Let $a_k = \frac{5k^2-9}{k^2-4}$ for $k = 3, 4, 5, \dots$. Prove that the sequence $\{a_k\}_{k=3}^{\infty}$ converges to 5 using the ϵ-N definition.

(f) Let $a_k = \frac{1}{k^2+3}$ for k natural numbers. Prove that $\lim_{k\to\infty} a_k = 0$ using the ϵ, N definition.

(g) Let $a_k = \frac{3k+2}{2k-1}$ for $k = 1, 2, 3, \dots$. Prove that $\lim a_k = \frac{3}{2}$ using the ϵ, N definition.

(h) Let $a_k = \frac{k^2+2}{k^2-3}$ for k natural numbers. Show that $\{a_n\}$ converges to 1 using the ϵ,N definition.

5. Write the statement of each of the following as stated in Stewart: the geometric series test/theorem(Sec 11.2); the divergence test (Sec 11.2, either box no. 6 or 7); the limit comparison test (Sec 11.4); the ratio test for positive terms (Sec 11.6); the alternating series test/theorem (Sec 11.5).

6. The following questions ask you to determine the converge/divergence of a series. To receive credit, give detailed explanation.
   (a) Determine whether the series $\sum_{k=1}^{\infty} \frac{k^k}{7^k(k)!}$ is convergent.
   (b) Determine whether the series $\sum_{k=1}^{\infty} \frac{k^k}{2^k(k)!}$ is convergent.
   (c) Determine whether the series $\sum_{n=3}^{\infty} \frac{6}{n\sqrt{n^2-8}}$ converges or diverges.
   (d) Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3-4n+2}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   (e) Determine whether the series $\sum_{n=1}^{\infty} \frac{n!}{7^n(n+8)!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   (f) Determine whether the series $\sum_{n=1}^{\infty} \frac{(n+8)!}{7^n n!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   (g) Determine whether the series $\sum_{n=1}^{\infty} \ln\left(\frac{3n}{n+1}\right)$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   (h) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+3}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   (i) Determine the convergence of $\sum_{k=1}^{\infty} k \cos\left(\frac{\pi k+1}{2k}\right)$.
   (j) Determine the convergence of $\sum_{k=1}^{\infty} \left(\frac{2k}{5k+5} + \frac{1}{(4)^k}\right)$
   (k) Determine the convergence of $\sum_{k=1}^{\infty} \frac{2^{4k+1}}{5^{2k-1}}$. If this series is convergent, compute its sum.
   (l) Determine the convergence of $\sum_{k=2}^{\infty} \frac{8^{3k+1}}{9^{2k-1}}$. If this series is convergent, compute its sum.
   (m) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges.
   (n) Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ converges.
7. Review pg 4-5 of `https://egunawan.github.io/fall18/notes/notes4_4lhopitals_rule.pdf`
   (a) Compute $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{3n}$. Compute $\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{n}$. Compute $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{5n}$.
   (b) Compute $\lim_{n\to\infty} n^2 e^{-n}$. Compute $\lim_{n\to\infty} \frac{\ln n}{n}$.
   (c) Compute $\lim_{n\to\infty} \frac{n \sin n}{n^2+1}$.
8. Consider the series $\sum a_n = \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n-1}$.
   (a) What are the first three terms in the series? ____________
   (b) Is the series convergent? You must justify. ____________

(c) Is the series $\sum a_n = \sum_{n=2}^{\infty} \frac{|\cos(n\pi)|}{n-1}$ convergent? You must justify. ______________

9. (a) Define an alternating series.
   (b) State the alternating series test.
   (c) Determine whether the series
   $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n}$$
   is convergent or divergent.
   (d) Determine whether the series $\sum_{k=1}^{\infty} \frac{(-1)^n\ 3n}{4n-1}$ is convergent or divergent.
   (e) Determine whether the series $\sum_{k=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ converges or diverges.

10. (a) Evaluate $\lim_{n\to\infty} e^{-n}\sqrt{n}$.
    (b) Determine whether $\sum_{n=0}^{\infty} e^{-n}\sqrt{n}$ converges or diverges.
    (c) Evaluate $\lim_{n\to\infty} \frac{(\ln(n))^2}{n^2}$.
    (d) Determine whether $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^2}$ converges or diverges.
    (e) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with the property that $a_1 + a_2 + \cdots + a_n = 2 - 3(0.8)^n$. State whether $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, find its sum.

11. Determine whether each series converges or diverges.

    a.) $\sum_{n=1}^{\infty} \frac{\ln\ n}{n}$ *b.)* $\sum_{n=4}^{\infty} \frac{1}{2^n - 9}$ *i.)* $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ *ii.)* $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

2 Part B: Integration techniques from Section 7.2, 7.3 and u-substitution

12. (Sec 7.2 Calculus and Trig Identities)
    (a) Show the formula $\sin(2x) = 2\sin x\cos x$ implies $\cos(2x) = \cos^2 x - \sin^2 x$ using differentiation: differentiate each identity and simplify to turn the first formula into the second.
    (b) Show the formula $\cos(2x) = \cos^2 x - \sin^2 x$ implies $\sin(2x) = 2\sin x\cos x$ using differentiation: differentiate each identity and simplify to turn the first formula into the second.
    (c) Use Pythagorean theorem and definitions of cos and sin to explain that $\sin^2\theta + \cos^2\theta = 1$.
    (d) Given that $\sin^2\theta + \cos^2\theta = 1$, show that $\tan^2\theta + 1 = \sec^2\theta$.
    (e) Given that $\sin^2\theta + \cos^2\theta = 1$ and $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, show that $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.
    (f) Given that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$, show that $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$.

13. (From class handouts)
    (a) Evaluate $\int_0^{\pi} \sin^3(5x)\,dx$.
    (b) Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$.
    (c) Evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$.

(d) (Note: You may use the fact sheets to look up derivatives and integrals of trig functions)
Evaluate 1.) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$ or 2.) $\int \frac{\cos^5 x}{\sin x} \, dx$ or 3.) $\int_0^{\pi} \cos^4(2x) \, dx$ or
4.) $\int \sin^3 x \cos^5 x \, dx$ or 5.) $\int \sin^2 x \cos^2 x \, dx$ or 6.) $\int \tan^3 x \sec^3 x \, dx$ or
7.) $\int \tan^2 x \sec^4 x \, dx$ or 8.) $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$ or 9.) omitted or 10.) $\int \tan^3 x \sec^4 x \, dx$.

(e) Evaluate 1.) $\int \frac{1}{x\sqrt{4-x^2}} \, dx$ or 2.) $\int \frac{1}{\sqrt{x^2+16}} \, dx$ or 3.) $\int_{\sqrt{2}}^{2} \left(\frac{1}{x^3\sqrt{x^2-1}}\right) dx$ or
4.) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}} \, dx$ or 5.) $\int_0^1 \left(\frac{1}{\sqrt{-x^2+2x+3}}\right) dx$ (Hint: First complete the square) or
6.) $\int \frac{1}{\sqrt{1+16x^2}} \, dx$ or 7.) $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} \, dx$ or 8.) $\int \frac{1}{x^2\sqrt{9x^2-1}} \, dx$ or
9.) $\int \sqrt{5+4x-x^2} \, dx$. (Hint: First complete the square)

14. Use either u-substitution or the techniques from Sec 7.2 and 7.3 to evaluate the following.

(a) $\int \frac{2x-3}{8+x^2} \, dx$

(b) $\int \frac{\sin(\ln(x))}{x} \, dx$

(c) Evaluate $\int_0^1 x \, e^{-x^2}$. Then perform a reality check against your answer.

(d) Evaluate $\int \frac{x}{\sqrt{4-x^2}}$. Prove your answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(e) True of false? Let $a > 0$. $\int_0^a f(x) = \int_0^a f(a-x) \, dx$. If T, justify. If F, give a counterexample. **T** **F**

(f) True of false? Let $a > 0$. $\int_0^a f(x) = \int_0^a f(x-a) \, dx$. **T** **F**
If T, justify. If F, give a counterexample.

(g) Write a sanity-check-type calculation (different from what you've written above) to further confirm your answer in the previous two questions.

(h) Evaluate $\int \frac{1}{x\ln(x)} \, dx$ ____________________

(i) Evaluate $\int \frac{1}{x^2+2x+4} \, dx$ ____________________

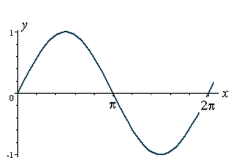
(j) Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.

15. (a) Sketch the graph of the function and shade the region whose area is represented by the integral $\int_{-3}^{4}(2x+15) - x^2 \, dx$. **Label all pertinent information.** Do not evaluate.

(b) Consider the region bounded by $y = x^2$, $y = 2 - x^2$.

i. Find the intersection points of the two curves.
ii. Sketch the two curves and shade the region bounded by the two curves.
iii. Set up, but **do not evaluate** an integral for the area of the shaded region.

(c) For your convenience, the graph of $y = \sin(x)$ is shown below.



Set up the definite integral for the area of the region bounded by the curves $y = \sin(x)$, $y = 0$, $x = 0$ and $x = \frac{\pi}{2}$. Then evaluate the area.