Name: ______________________________ **Math 1152Q: Fall '17**
Week 9 Sample Quiz

Summary: 11.10 intro to taylor series, multiplication and division of power series.

1. (a) *If* f has a power series representation at 4, that is, if $f(x) = \sum_{n=0}^{\infty} c_n (x-4)^n$ for $|x-4| < R$, then its coefficients are given by the formula $c_n =$ ________________. Answer: Theorem 5 on page 760.

   (b) Explain briefly how to prove your formula for c_n. Answer: follow your self-study notes from pg 759, or see lecture notes `https://egunawan.github.io/fall17/notes/notes11_10part1.pdf`.

   (c) When is a function $f(x)$ equal to the sum of its Taylor series? See explanation on pg 761. Either state one of the equations on bottom half of pg 761 or Theorem 8.

2. Circle all the true statements and cross out all the false statements. You should be able to justify each to yourself, but you don't need to write down explanations.

   (a) If f has derivatives of all orders, then f can be represented as a power series at $x = 0$. See Note, middle of pg 760.

   (b) There exists a function which is not equal to its Maclaurin series. See Note, middle of pg 760.

   (c) If the series $\sum_{n=1}^{\infty} c_n x^n$ converges for $|x| < R$, then $\lim_{n\to\infty} c_n x^n = 0$ for $|x| < R$. See explanation, first sentence of pg 763. Or see Sec 11.2, Thm 6, pg 713.

   (d) If the series $\sum_{n=1}^{\infty} c_n x^n$ diverges for $x = 5$, then $\lim_{n\to\infty} c_n x^n \neq 0$ for $x = 5$. A counterexample: $c_n = \frac{1}{n\,5^n}$. See Ex. 9 Sec 11.2, pg 713.

3. (11.10 multiplication and division of power series pg 770 - see class notes on Wed)

   (a) By multiplying and dividing power series, write a (simple) function $g(x)$ with a power series expansion whose coefficients are the Fibonacci numbers, that is, $g(x) = 0+1x+1x^2+2x^3+3x^4+5x^5+8x^6+\ldots$. This function $g(x)$ is called the *generating function* for the sequence $\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$.

   J. Fowler video: `https://www.coursera.org/learn/advanced-calculus/lecture/X5UhL/what-is-a-formula-for-the-fibonacci-numbers`

   (b) Find the radius of convergence R. Answer: Using the ratio test, you can show that $\boxed{R = 2/(1+\sqrt{5})\text{, the reciprocal or the golden ratio}}$

4. (11.10 applying Maclaurin series definition - see class notes on Mon)

   Consider catenary bridges and arches (`https://en.wikipedia.org/wiki/Catenary`) or the shape of a hanging rope.

   (a) What function describes their shapes? Write it using exponential functions.

   (b) Using the table `http://egunawan.github.io/fall17/quizzes/11_10_table01.pdf` (printed on the next page) or by following Ex. 2 pg 763, write a power series representation for e^x and e^{-x}.

   (c) Using the previous part, write a power series representation for the function describing the shapes of the Catenary bridges.

5. (Section 11.10 WebAssign)

   (a) (no 1) Find the Maclaurin series for $f(x) = 6(1-x)^{-2}$ using the definition of a Maclaurin series. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence.

   Answer: Use Taylor series theorem/formulas 5,6,7 on pg 760. Follow Example 8 but replace $(1+x)^k$ with $(1-x)^{-2}$. The Maclaurin series is $\boxed{\sum_{n=0}^{\infty} 6(n+1)x^n}$. Use Ratio Test to find the radius of convergence $\boxed{R = 1}$.

   (b) (no 2) Find the Maclaurin series for $f(x) = ln(1+5x)$. Don't use the table. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence.

   Answer: Use Taylor series theorem/formulas 5,6,7 on pg 760. The Maclaurin series is $\boxed{f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(5x)^n}{n}.\ R = 1/5 \text{ by the Ratio Test.}}$

(c) (no 3) Use a Maclaurin series given in this table `http://egunawan.github.io/fall17/quizzes/11_10_table01.pdf` (printed on the next page) to obtain the Maclaurin series for the function $f(x) = 8e^x + e^{8x}$. Find the radius of convergence.

Answer: Use the table to get $e^x = \sum_{n=1}^{\infty} x^n n!$. Apply the Composition Theorem with $h(x) = 8x$ and $f(t) = e^t$ to get $e^{8x} = \sum_{n=1}^{\infty} \frac{(8x)^n}{n!}$. Apply 'sum' theorem for series (pg 714 Sec 11.2) to get the sum $\boxed{\sum_{n=0}^{\infty} (8+8^n)\frac{x^n}{n!}. \text{ The series is convergent for all real numbers}}$.

(d) (no 4) Evaluate the indefinite integral $\left(8 \int \frac{e^x - 1}{5x} \, dx\right)$ as an infinite series.

Answer: Read Example 11 pg 768-769 for similar problem. This answer gives the Maclaurin series but you can choose a different Taylor series centered not at 0. First either use the table or directly evaluate the Maclaurin series for $e^x - 1 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$. Multiply this Maclaurin series by $\frac{1}{x}$ to get $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$. Apply term-by-term integration to get final answer, $\boxed{\frac{8}{5}\sum_{n=1}^{\infty} \frac{x^n}{(n)n!} + C}$.

(e) (no 5) Use series to approximate the definite integral $I = \int_0^{0.5} x^2 e^{-x^2} \, dx$ so that error is smaller than 0.001. (You may leave the estimate as a partial sum without simplifying).

Answer: Follow Example 11a pg 769 to evaluate $e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$. Multiply this series by x^2 to get the Maclaurin series for $x^2 e^{-x^2}$. Apply term-by-term integration to get the series representation for the indefinite integral $\int x^2 e^{-x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(2n+3)} + C$. The series representation for the given definite integral is $\boxed{\int_0^{0.5} x^2 e^{-x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n 0.5^{2n+3}}{n!(2n+3)}}$.

The term with $n = 2$ is equal to $1/(2^7 \; 2! \; 7) = 1/\left((2^8)(7)\right) = 1/((256)(7))$ which is smaller than $1/1000$. So, by Alternating Series Estimate Theorem (sec 11.5), $\boxed{\text{the partial sum approximation using the first two terms, } \sum_{n=0}^{1} \frac{(-1)^n 0.5^{2n+3}}{n!(2n+3)}}$ is enough to guarantee that the error is less than 0.001.

(f) (no 6) Find the Maclaurin series for $f(x) = e^{-4x}$ using the definition of a Maclaurin series. Don't use the table. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence R.

Answer: Follow Example 1 pg 760 but replace x with $-4x$ You get $\boxed{e^{-4x} = \sum_{n=0}^{\infty} \left(\frac{(-4)^n}{n!}\right) x^n. \text{ The series is convergent for all real numbers}}$.

The following table will be provided on the quiz.

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$