Name: ______________________________

Math 1152Q: Fall '17
Week 8 Sample Quiz

Summary: 11.3: Integral test and p-series concepts; 11.8: using geometric series test or ratio test to find the interval of convergence of a series; 11.9: find a power series representation of a function and determining the radius of convergence.

1. (Section 11.3 concept)
   (a) Suppose f is a continuous, positive, and decreasing function on $[1, \infty)$ and $a_k = f(k)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\ \mathrm{dx} \qquad \sum_{k=1}^{5} a_k \qquad \sum_{k=2}^{6} a_k$$

   (b) For each statement, determine whether it's true or false and give a brief justification:
      1. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test.
      2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ diverges by the p-series test.
      3. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
      4. The exact sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is an open question.
      5. $\int_5^{\infty} \frac{1}{x^2}\ \mathrm{dx} = \frac{1}{5}$
      6. $\int_5^{\infty} \frac{1}{x^2}\ \mathrm{dx} = 5$

      (Answer: see Sec 11.3, page 722)

   (c) Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq N$. Suppose $\sum_{n=1}^{\infty} a_k = S$. Let $S_N := a_1 + a_2 + \cdots + a_N$.
      Which error estimate gives the more accurate approximation?
      1. $\int_{n+1}^{\infty} f(x)\ \mathrm{dx} \leq S - S_N \leq \int_n^{\infty} f(x)\ \mathrm{dx}$
      2. $S_N + \int_{n+1}^{\infty} f(x)\ \mathrm{dx} \leq S \leq S_N + \int_n^{\infty} f(x)\ \mathrm{dx}$

      (Answer: see Sec 11.3, page 724)

   (d) Is the following statement true or false? Justify. (See answer at the end of the file).
      Suppose $f(x)$ is a continuous function defined on $[5, \infty)$. If $f(x)$ is not bounded on $[5, \infty)$, we cannot apply the integral test using $\int_5^{\infty} f(x)\ \mathrm{dx}$. .

2. (Sec 11.3 more p-series questions)
   (Note: the symbol ζ is the lower-case Greek letter which is pronounced "zeta" in English).
   The *Riemann zeta*-function ζ is defined by

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

   It is used in number theory to study the distribution of prime numbers.
   (a) What is the domain of the function ζ? (That is, for what values of x is this function defined?)
      (Hint: go to Sec 11.3, page 722)
   (b) Euler computed $\zeta(2)$ to be $\frac{\pi^2}{6}$. (See page 720, sec 11.3). Use this fact to find the sum of each series below.

$$\sum_{n=3}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(5n)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

3. (Section 11.8 power series)
   (a) What is a power series? (See Sec 11.8, top of page 747)
   (b) What is the radius of convergence of a power series? (There are three cases. See Sec 11.8, top of page 749)
   (c) In most cases, how do you find the radius of convergence of a power series? (See the test used in Examples 1-5 in Sec 11.8, pg 747-750)
   (d) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}. \qquad \text{Answer: see Example 5, pg 750.}$$

   (e) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} n! x^{2n}. \qquad \text{Answer: see Example 1, pg 747.}$$

   (f) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n^5}. \qquad \text{Answer: same radius of convergence as Example 2, pg 747, but both endpoints are included.}$$

   (g) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}. \qquad \text{Answer: same answer as Example 3, pg 748.}$$

4. (Sec 11.9 WebAssign finding interval of convergence) For each function, find a power series representation and determine the interval of convergence.
   (Check your work with WolframAlpha. Type "series representation of ...")
   (a) $f(x) = \dfrac{1}{3+x}$ (see Sec 11.9 Example 2)
   (b) $f(x) = \dfrac{x^3}{5+x}$ (see Sec 11.9 Example 3)
   (c) $f(x) = \dfrac{x}{1+10x^2}$ (a variation of Sec 11.9 Example 3)

5. (Sec 11.9 WebAssign differentiation and integration of power series) For each function, find a power series representation. Determine the radius of convergence. (You do not need to determine the interval of convergence - only the radius of converge because of Note 2 pg 754).
   (a) $f(x) = \dfrac{1}{(2+x)^2}$ (a variation of Sec 11.9 Example 5)
   (b) $f(x) = \ln(1+x)$ (see Sec 11.9, Example 6)
   (c) $f(x) = arctan(x)$ (see Sec 11.9, Example 7)
   (d) $\displaystyle\int \frac{1}{1+x^7}$ dx (see Sec 11.9, Example 8)
   (e) $\displaystyle\int \frac{x}{1-x^7}$ dx (a variation Sec 11.9, Example 8)

6. (a couple answers)
   - Answer to 1(d) is True. Justification: If $g(x)$ is positive and decreasing on $[5,\infty)$, then it is bounded (for example, bounded below by 0 and bounded above by $g(5)$). The contrapositive of this statement is: If $g(x)$ is not bounded, then it is not positive or not decreasing. Since $g(x)$ does not meet at least one of the criteria for applying the integral test using $\int_5^\infty g(x)\,dx$, we cannot apply the integral test using $\int_5^\infty g(x)\,dx$.
   - Answer to 3(c): ratio test usually works.