Name (please print): KEY

Math 1152Q: Fall '17
Week 4 Quiz KEY, Friday, Sept 22, 2017

You may use the fact that

$$\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e \quad \text{and} \quad \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \frac{1}{e}$$

without explanation. Note that the first few digits of e are 2.718.

1. (a) Complete the statement of the *ratio* test:
   Suppose a_n is a positive number for all $n \geq 1$.
   ANSWER: copy from Sec 11.6. Don't forget to include $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ **in your statement.**

   (b) Please circle **ONE** series that you want to work with:

$$\sum \frac{5^n}{n!} \quad \text{or} \quad \sum \frac{n!}{n^n} \text{ (similar to Sec 11.6 Example 5 pg 741).}$$

   (c) Show your work for attempting to use the ratio test to this series to determine whether it converges or diverges.

For $a_n = \frac{5^n}{n!}$:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)!}\frac{n!}{5^n}\\ &= \lim_{n\to\infty}\frac{5}{n+1}\\ &= \lim_{n\to\infty}\frac{\frac{5}{n}}{1+\frac{1}{n}}\\ &= \frac{\lim_{n\to\infty}\frac{5}{n}}{1+\lim_{n\to\infty}\frac{1}{n}}\\ &= 0.\end{aligned}$$

For $a_n = \frac{n!}{n^n}$:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{(n+1)}}\frac{n^n}{n!}\\ &= \lim_{n\to\infty}(n+1)\frac{n^n}{(n+1)^{(n+1)}}\\ &= \lim_{n\to\infty}(n+1)\frac{n^n}{(n+1)^n(n+1)}\\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\\ &= \frac{1}{e} \text{ as given at the top of the page}\end{aligned}$$

See also book's solution for a similar-looking series Sec 11.6 Example 5 pg 741.

   (d) If the ratio test is conclusive, determine whether the series is convergent or divergent. Otherwise, state that the ratio test is inconclusive.

The two series converge by the ratio test since $0 < 1$ and $\frac{1}{e} < 1$.

2. Please *circle* **ONE** series below to work with:

$$\sum_{n=1}^{\infty} \frac{\ln\ n}{n} \quad \text{or} \quad \sum_{n=4}^{\infty} \frac{1}{2^n - 9} \text{ (from Sec 11.4 Examples 2 and 3).}$$

Try Divergence Test: Both $\frac{\ln\ n}{n}$ and $\frac{1}{2^n-9}$ converge to 0. So the Divergence Test is inconclusive.

Try Ratio test:
For $a_n = \frac{\ln\ n}{n}$**:** you know that ratio tests will be *inconclusive* for terms involving only constants, log, and polynomials. See first page of "growth rates" notes: `https://egunawan.github.io/fall17/notes/notes11_6part3key.pdf`

For $a_n = \frac{1}{2^n-9}$**:**

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{2^n - 9}{2^{n+1} - 9} \\ &\overset{L}{=} \lim_{n\to\infty} \frac{\ln 2\ 2^n}{\ln 2\ 2^{n+1}} \\ &= \frac{1}{2} < 1.\end{aligned}$$

By the ratio test, $\sum_{n=4}^{\infty} \frac{1}{2^n-9}$ converges.

Try Limit Comparison Test (LCT):
For $a_n = \frac{\ln\ n}{n}$**:** Try $b_n = \frac{1}{n}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n}\frac{n}{1} \\ &= \lim_{n\to\infty} \ln n = \infty.\end{aligned}$$

Since $\sum b_n$ diverges by the harmonic series test (p-series test), by the Limit Comparison Test, we conclude that $\sum a_n$ also diverges. (See also book's solution using Comparison Test. Sec 11.4 Example 2 pg 728-729.)

For $a_n = \frac{1}{2^n-9}$**:** Try $b_n = \frac{1}{2^n}$

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1}{2^n - 9}\frac{2^n}{1} \\ &= \lim_{n\to\infty} \frac{2^n}{2^n - 9} \\ &= \lim_{n\to\infty} \frac{1}{1 - \frac{9}{2^n}} \\ &= \frac{1}{1 - \lim_{n\to\infty} \frac{9}{2^n}} = 1\end{aligned}$$

We know that $\sum b_n$ converges by the geometric series test (or you can show this using the ratio test or root test). So, by the Limit Comparison test, we conclude that $\sum a_n$ also diverges. (Warning: if $\lim_{n\to\infty} \frac{a_n}{b_n}$ had been ∞, we cannot conclude anything).

3. Please *circle* **ONE** series below to work with:

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \text{ (from Sec 11.4 Examples 1 and 4).}$$

Try Divergence Test: Both $\frac{2n^2+3n}{\sqrt{5+n^5}}$ and $\frac{5}{2n^2+4n+3}$ converge to 0. The Divergence Test is inconclusive.

Try Ratio test (for both): Before trying it, you know that ratio tests will be inconclusive for series that look like p-series.

Try Limit Comparison Test (LCT):
For $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$**:**
Try $b_n = \frac{1}{\sqrt{n}}$.
See textbook Sec 11.4 Example 4 page 730 for solution.

For $a_n = \frac{5}{2n^2+4n+3}$**:**
Try $b_n = \frac{1}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{5}{2n^2+4n+3}\frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{5n^2}{2n^2+4n+3} \\
&= \lim_{n\to\infty} \frac{5}{2+\frac{4n}{n^2}+\frac{3}{n^2}} \\
&= \frac{5}{2+\lim_{n\to\infty}\frac{4}{n}+\lim_{n\to\infty}\frac{3}{n^2}} \\
&= \frac{5}{2+0+0} \\
&= \frac{5}{2}.
\end{aligned}$$

Since $\sum b_n$ converges by the p-series test, by the Limit Comparison Test, we conclude that $\sum a_n$ also converges.

(See also textbook Sec 11.4 Example 1 page 728 for solution using the comparison test. I think my LCT solution will takes less time to do in case the term of your series has minus signs in the denominator.)