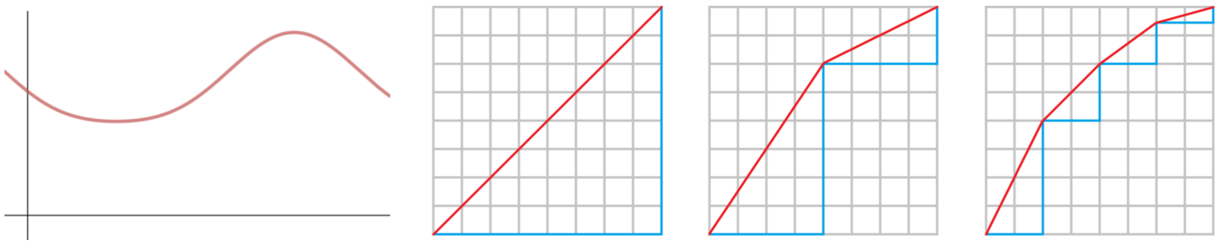
Arc Length

Arc Length

Suppose that a curve C is defined by the equation $y = f(x)$, where f is **continuous** and $a \le x \le b$. We obtain a **polygonal approximation** to C by dividing the interval $[a,b]$ into n subintervals with endpoints x_0, x_1, , x_n and equal width Δx. If $y_i = f(x_i)$, then the point P_i (x_i, y_i) lies on C and the polygon with vertices P_0, P_1, , P_n is an approximation to C.



The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. Therefore, we define the **length** L of the curve C with equation $y = f(x)$, $a \le x \le b$, as the limit of the lengths of these inscribed polygons (provided the limit exists).

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} | P_{i-1}P_i |$$

The definition of arc length given above is not very convenient for computational purposes, but we can derive an integral formula for L in the case where f has a continuous derivative. Such a function f is called **smooth** because a small change in x produces a small change in $f'(x)$.

If we let $\Delta y_i = y_i - y_{i-1}$, then

$$\begin{aligned} | P_{i-1}P_i | &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \\ &= \sqrt{(\Delta x)^2 \left[1 + \left(\frac{\Delta y_i}{\Delta x} \right)^2 \right]} \\ &= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x} \right)^2} \cdot \Delta x \end{aligned}$$

By applying the **Mean Value Theorem** to f on the interval $[x_{i-1}, x_i]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

Thus we have

$$|P_{i-1}P_i| = \sqrt{1+\left(\frac{\Delta y_i}{\Delta x}\right)^2}\cdot \Delta x$$
$$= \sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\cdot \Delta x$$

Therefore,

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i|$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\cdot \Delta x$$
$$= \int_a^b \sqrt{1+\left[f'(x)\right]^2}\, dx$$

The integral exists because the integrand $\sqrt{1+\left[f'(x)\right]^2}$ is continuous. Thus we have the following theorem.

Formula **The Arc Length formula**

If f' is continuous on $[a,b]$, then the length of the curve $y=f(x)$, $a\le x\le b$, is

$L =$ ____.

Similarly, if a curve has the equation $x=g(y)$, $c\le y\le d$, and $g'(y)$ is continuous, then

$L =$ ____.

Example:

Find the length of the curve $y^2=x^3$ for $1\le x\le 4$.

The Arc Length Function

If a smooth curve C has the equation $y = f(x)$, $a \le x \le b$, let $s(x)$ be the distance along C from the initial point $P_0\left(a, f(a)\right)$ to the point $Q\left(x, f(x)\right)$. Then s is a function, called the arc length function.

$$s(x) = \int_a^x \sqrt{1 + \left[f'(t)\right]^2}\, dt$$

Special Problem

Example:

Find the length of the curve $f(x) = x^3 + \frac{1}{12x}$ on the interval $\left[\frac{1}{2}, 2\right]$.