Improper Integrals

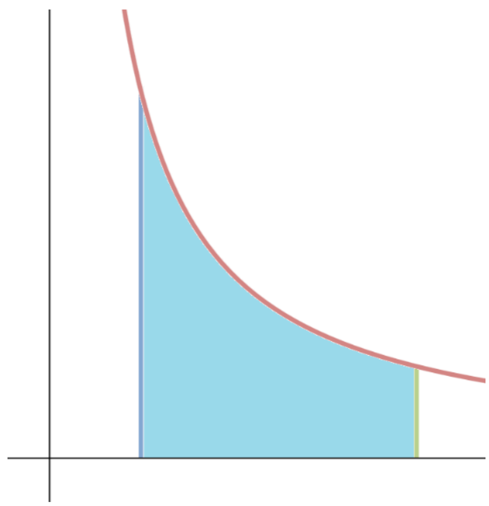
Idea

The **improper integral** is used for cases in which

- The **interval** of integration is **infinite** or
- The **integrand** has an **infinite discontinuity** on the interval of integration.

Infinite Intervals

Consider the integral $\int_1^b \frac{1}{x^2}\,dx$, for any real number $b > 1$.



$$\int_1^b \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)_1^b = 1 - \frac{1}{b}$$

$b = 2$	$b = 3$	$b = 4$		$b \to \infty$
$1 - \frac{1}{2}$	$1 - \frac{1}{3}$	$1 - \frac{1}{4}$		

We express this result as

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

which is an improper integral because ∞ appears in the upper limit.

Improper Integrals

Definitions **Improper Integrals over Infinite Intervals**

1. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \underline{\hspace{8cm}},$$

   provided this limit exists (as a finite number).

2. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \underline{\hspace{8cm}},$$

   provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called

- **convergent** if the corresponding limit exists and
- **divergent** if the limit does not exist.

3. If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \underline{\hspace{10cm}}.$$

Example:

Evaluate $\int_0^\infty \frac{1}{1+x^2}\,dx$.

Example:

Evaluate $\int_0^{\infty} \cos x \, dx$.

Example:

Evaluate $\int_2^{\infty} \frac{1}{x^2} \cos\left(\frac{\pi}{x}\right) dx$.