USING GROWTH RATES TO DETERMINE WHETHER A SERIES CONVERGES.

Refer to the "growth rates" notes.

1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\ln\ (n+1)}$$

(a) The denominator of the term is $ln\ (n+1)$. Consider the function $\ln(x+1)$. Fill in the blank with $\ll$ or $\gg$:

$$\ln(x+1) \ \underline{\ll}\ \ x.$$

(b) This means that

$$\lim_{x\to\infty} \frac{x}{\ln(x+1)} = \infty$$

(See the "growth rates" notes)

(c) Fill in the blank with $\leq$ or $\geq$:

$$\ln(n+1) \ \underline{\leq}\ \ n \quad \text{for large enough } n.$$

(d) Fill in the blank with $\leq$ or $\geq$:

$$\frac{1}{\ln(n+1)} \ \underline{\geq}\ \ \frac{1}{n} \quad \text{for large enough } n.$$

(e) State the test which you would use to determine whether the series $\sum \frac{1}{n}$ converges or diverges.

p-series test or harmonic series test

(f) State whether the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

$\sum \frac{1}{n}$ diverges

(g) State the statement of the limit comparison test.

(h) Compute $\lim_{n\to\infty} \frac{n}{\ln(n+1)}$. Note that the answer is given in part (b).

$$\lim_{n\to\infty} \frac{n}{\ln(n+1)} \overset{L}{=} \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n+1}\right)} = \lim_{n\to\infty} n+1 = \infty$$

(i) By the limit comparison test and part (f), the series

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} \ \underline{\text{diverges}}$$

2. Let p be any positive integer (say, $p = 5$) Let a be a number larger than 1 (say, $a = \frac{3}{2}$). Consider the series

$$\sum_{n=1}^{\infty} \frac{n^p}{a^n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n}.$$

After simplifying, we realize that the term of the series is

$$\frac{n^5}{(\frac{3}{2})^n}.$$

(a) The series $\sum_{n=1}^{\infty} n^5$ diverges by the p-series OR divergence test.

(b) The series $\sum_{n=1}^{\infty} (\frac{2}{3})^n$ Converges by the geometric series test.

(c) Let's figure out: is the numerator/ denominator more dominant than the other?

(d) The denominator of the term is $\left(\frac{3}{2}\right)^n$. Consider the function $\left(\frac{3}{2}\right)^x$. exponential
The numerator of the term is n^5. Consider the function x^5. polynomial
Fill in the blank with $\ll$ or $\gg$:

$$x^5 \quad \ll \quad \left(\frac{3}{2}\right)^x.$$

(e) This means that (check the "growth rates" notes)

$$\lim_{x\to\infty} \frac{\left(\frac{3}{2}\right)^x}{x^5} = \infty. \text{ This also means } \lim_{x\to\infty} \underline{\hspace{3em}} = 0.$$

(f) Part (e) means that $\sum \left(\frac{2}{3}\right)^n$ is more dominant than $\sum n^5$.

(g) Quiz your classmate or yourself on the statement of the ratio test until it's memorized.

(h) (I told you that when you see *only* polynomial-like terms, like $n^{p_1} + n^{p_2}$, the ratio test *will be inconclusive* (convince yourself). But, if you see powers like a^n it's OK to use the ratio test. Let's apply the ratio test to $a_n = \frac{n^5 2^n}{3^n}$. Compute

$$\lim_{n\to\infty} \frac{(n+1)^5}{\left(\frac{3}{2}\right)^{(n+1)}} \frac{\left(\frac{3}{2}\right)^n}{n^5}. = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^5 \frac{2}{3}$$

because x^5 is continuous at 1

$$= \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^5 \frac{2}{3}$$

$$= 1^5 \cdot \frac{2}{3} = \frac{2}{3} < 1$$

(i) By the ratio test, the series

$$\sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n} \quad \text{Converges}.$$

:)

3. Consider the series

$$\sum_{n=1}^{\infty} a_n$$

for

$$a_n = \frac{n^n}{7^n\,(n)!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\,(n)!}$$

(a) Look at the term of the series.

The numerator, n^n looks like the function $\underline{\ x^x\ }$.

The denominator, $7^n\,(n)!$ looks like the product of functions $\underline{\ 7^x\ }$ and $\underline{\ x!\ }$.

(b) Which is more dominant for large n? The numerator or the denominator? Can you tell just by looking at the "growth rates" notes?

convergent $\rightarrow$ $\sum \frac{1}{7^n}\frac{1}{n!}$ and $\sum n^n$ $\leftarrow$ divergent

(c) I told you that the ratio test *will probably work* if you see exponents like r^n or a factorial. Compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

for

$$a_n = \frac{n^n}{7^n\;n!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\;n!}$$

(Use the fact that $(1+\frac{1}{n})^n \to e$ as $n\to\infty$)

Answer Let $a_n = \dfrac{n^n}{r^n\,n!}$ where r is a positive number.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{r^{n+1}(n+1)!}\;\frac{r^n\,n!}{n^n} = \frac{1}{r(n+1)}\;\frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{r}\left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{r}\left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{r}\left(1+\frac{1}{n}\right)^n = \frac{1}{r}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{e}{r}$$

(d) By the ratio test, the series

$$\sum \frac{n^n}{7^n\;n!}\ \underline{\text{ converges }} \qquad \text{and} \qquad \sum \frac{n^n}{2^n\;n!}\ \underline{\text{ diverges }}.$$

because $\frac{e}{7} < 1$ because $\frac{e}{2} > 1$