USING GROWTH RATES TO DETERMINE WHETHER A SERIES CONVERGES.

Refer to the "growth rates" notes.

1. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{\ln\ (n+1)}$$

   (a) The denominator of the term is *ln* $(n+1)$. Consider the function $\ln(x+1)$. Fill in the blank with $\ll$ or $\gg$:
   $$\ln(x+1) \quad \underline{\qquad} \quad x.$$

   (b) This means that
   $$\lim_{x\to\infty} \underline{\qquad\qquad} = \infty$$
   (See the "growth rates" notes)

   (c) Fill in the blank with $\leq$ or $\geq$:
   $$\ln(n+1) \quad \underline{\qquad} \quad n \qquad \text{for large enough } n.$$

   (d) Fill in the blank with $\leq$ or $\geq$:
   $$\frac{1}{\ln(n+1)} \quad \underline{\qquad} \quad \frac{1}{n} \qquad \text{for large enough } n.$$

   (e) State the test which you would use to determine whether the series $\sum \frac{1}{n}$ converges or diverges.

   (f) State whether the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

   (g) State the statement of the limit comparison test.

   (h) Compute $\lim_{n\to\infty} \frac{n}{\ln(n+1)}$. Note that the answer is given in part (b).

   (i) By the limit comparison test and part (f), the series
   $$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} \quad \underline{\qquad\qquad\qquad}$$

2. Let p be any positive integer (say, $p = 5$) Let a be a number larger than 1 (say, $a = \frac{3}{2}$). Consider the series

$$\sum_{n=1}^{\infty} \frac{n^p}{a^n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n}.$$

After simplifying, we realize that the term of the series is

$$\frac{n^5}{(\frac{3}{2})^n}.$$

(a) The series $\sum_{n=1}^{\infty} n^5$ ______________ by the ____________________________________ test.

(b) The series $\sum_{n=1}^{\infty} (\frac{2}{3})^n$ ________________ by the ___________________________ test.

(c) Let's figure out: is the numerator/ denominator more dominant than the other?

(d) The denominator of the term is $\left(\frac{3}{2}\right)^n$. Consider the function $\left(\frac{3}{2}\right)^x$.
The numerator of the term is n^5. Consider the function x^5.
Fill in the blank with $\ll$ or $\gg$:

$$x^5 \quad \underline{\qquad} \quad \left(\tfrac{3}{2}\right)^x.$$

(e) This means that (check the "growth rates" notes)

$$\lim_{x\to\infty} \underline{\qquad\qquad} = \infty. \text{ This also means } \lim_{x\to\infty} \underline{\qquad\qquad} = 0.$$

(f) Part (e) means that ______________________ is more dominant than ___________________.

(g) Quiz your classmate or yourself on the statement of the ratio test until it's memorized.

(h) (I told you that when you see *only* polynomial-like terms, like $n^{p_1} + n^{p_2}$, the ratio test *will be inconclusive* (convince yourself). But, if you see powers like a^n it's OK to use the ratio test. Let's apply the ratio test to $a_n = \frac{n^5 2^n}{3^n}$. Compute

$$\lim_{n\to\infty} \frac{(n+1)^5}{\left(\frac{3}{2}\right)^{(n+1)}} \frac{\left(\frac{3}{2}\right)^n}{n^5}.$$

(i) By the ratio test, the series

$$\sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n} \quad \underline{\qquad\qquad\qquad}.$$

3. Consider the series

$$\sum_{n=1}^{\infty} a_n$$

for

$$a_n = \frac{n^n}{7^n\ (n)!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\ (n)!}$$

(a) Look at the term of the series.

The numerator, n^n looks like the function _______.

The denominator, $7^n\ (n)!$ looks like the product of functions ___________ and ____________.

(b) Which is more dominant for large n? The numerator or the denominator? Can you tell just by looking at the "growth rates" notes?

(c) I told you that the ratio test *will probably work* if you see exponents like r^n or a factorial. Compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

for

$$a_n = \frac{n^n}{7^n\ n!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\ n!}$$

(Use the fact that $(1+\frac{1}{n})^n \to e$ as $n \to \infty$)

(d) By the ratio test, the series

$$\sum \frac{n^n}{7^n\ n!} \ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad \text{and} \qquad \sum \frac{n^n}{2^n\ n!} \ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$