Growth Rates

Definition **Growth Rates of Functions (as *x* approaches infinite)**

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then

- f **grows faster than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.
- f and g **have comparable growth rates** if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$, where $0 < M < \infty$.

Theorem **Asymptotic Hierarchy**

Let $f \ll g$ mean that g grows faster than f as $x \to \infty$. Then

$$c \ll (\ln x)^q \ll x^p \ll a^x \ll x! \ll x^x$$

$$\lim_{n\to\infty} \sqrt[n]{c} \to 1 \qquad \lim_{n\to\infty} \sqrt[n]{(\ln n)^q} \to 1 \qquad \lim_{n\to\infty} \sqrt[n]{n^p} \to 1 \qquad \lim_{n\to\infty} \sqrt[n]{a^n} \to a \qquad \lim_{n\to\infty} \sqrt[n]{n!} \to \infty \qquad \lim_{n\to\infty} \sqrt[n]{n^n} \to \infty$$

(tricky)

Example:

Use the **Root Test** to determine whether the series $\sum_{k=2}^{\infty} \frac{3^k \ln k}{2^k k^5}$ converge. → See work below

$$\lim_{n\to\infty} c^{\frac{1}{n}} = c^{\left(\lim_{n\to\infty} \frac{1}{n}\right)} = c^0 = 1$$

c^x is continuous at 0

$$\lim_{n\to\infty} n^{\frac{p}{n}} = ?$$

Use L'hopital Rule to show

$$\lim_{n\to\infty} n^{\frac{p}{n}} = e^0 = 1$$

$$\lim_{n\to\infty} (\ln n)^{\frac{q}{n}} = ?$$

Let $y = (\ln x)^{\frac{q}{x}}$

$$\ln y = \ln\left[(\ln x)^{\frac{q}{x}}\right] = \frac{q}{x} \ln\left[(\ln x)\right]$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{9 \ln(\ln x)}{x}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{\left(9 \frac{1}{(\ln x)}\right)}{1}$$

$$= 0$$

So $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y}$

since e^x is continuous at 0 $\;= e^{\lim_{x\to\infty} \ln y}$

$$= e^0 = 1$$

For $a_k = \frac{3^k}{2^k} \frac{\ln k}{k^5}$, $\sum a_k$ converges or diverges?

Note that $\sum \frac{3^k}{2^k}$ diverges by geom series test

$\sum \ln k$ diverges by divergence test

$\sum \frac{1}{k^5}$ converges by p-series test.

But $\underbrace{x^5}_{\text{polynomial}} \ll \underbrace{\left(\frac{3}{2}\right)^x}_{\text{exponential}}$

so $\sum \left(\frac{3}{2}\right)^k$ wins, and $\sum a_k$ diverges.

Answer using root test:

$$\sqrt[k]{a_k} = \frac{3}{2} \frac{(\ln k)^{\frac{1}{k}}}{k^{\frac{5}{k}}}$$

$$\lim_{k \to \infty} (\ln k)^{\frac{1}{k}} = ?$$

Let $y = (\ln x)^{\frac{1}{x}}$

$$\ln y = \ln \left[(\ln x)^{\frac{1}{x}} \right]$$

$$= \frac{1}{x} \ln \left[(\ln x) \right]$$

$$\lim_{x \to \infty} \ln y \stackrel{L}{=} \frac{\left(\frac{1}{\ln x} \right)}{1} = 0$$

So, $\lim_{x \to \infty} y = \lim_{x \to \infty} e^{\ln y} \underset{\uparrow}{=} e^{\lim_{x \to \infty} \ln y} = e^0 = 1$

e^x is continuous at 0

So $\lim_{k \to \infty} (\ln k)^{\frac{1}{k}} = 1$

Similarly, $\lim_{k \to \infty} k^{\frac{5}{k}} = 1$

So $\lim_{k \to \infty} \sqrt[k]{a_k} = \frac{3}{2} > 1.$

By the root test,

$\sum a_k$ diverges.