The Integral Test and Estimates of Sums

Example:
Suppose $f(x)$ is a continuous and positive function on $[1,\infty)$.

a. Use the **Right Endpoint Rule** with $n=5$ to approximate the integral $\int_1^6 f(x)\,dx$.

b. Use the **Left Endpoint Rule** with $n=5$ to approximate the integral $\int_1^6 f(x)\,dx$.

c. Suppose $f(x)$ is **decreasing**, then (fill in <, = or >)

the estimated value in part (a) ____________ the value of $\int_1^6 f(x)\,dx$ and

the estimated value in part (b) ____________ the value of $\int_1^6 f(x)\,dx$.

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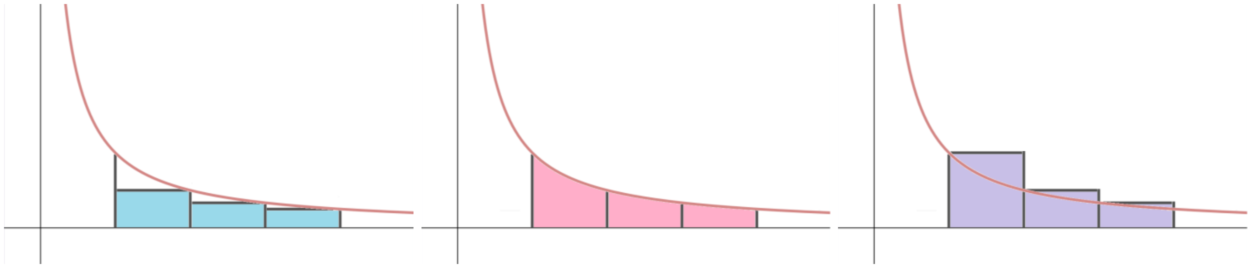
Integral Test

Suppose $f(x)$ is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then



$$a_2 + a_3 + a_4 \quad \le \quad \int_1^4 f(x)\,dx \quad \le \quad a_1 + a_2 + a_3$$

In general,

$$\sum_{k=2}^{n} a_k \;\le\; \int_1^n f(x)\,dx \;\le\; \sum_{k=1}^{n-1} a_k$$

The Integral Test

Suppose f is a **continuous**, **positive**, **decreasing** function on $[1,\infty)$ and let $a_n = f(n)$. Then

- If $\int_1^\infty f(x)\,dx$ is **convergent**, then $\sum_{n=1}^{\infty} a_n$ is ______________________ .
- If $\int_1^\infty f(x)\,dx$ is **divergent**, then $\sum_{n=1}^{\infty} a_n$ is ______________________ .

When we use the Integral Test

- It is not necessary to start the series or the integral at $n=1$. For example, in testing the series $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ we can use $\int_4^\infty \frac{1}{(x-3)^2}\,dx$.
- It is not necessary that f be **always** decreasing. What is important is that f be **ultimately** decreasing. That is, decreasing on $[N,\infty)$ for some number N. Then $\sum_{n=N+1}^{\infty} a_n$ is convergent, which means $\sum_{n=1}^{\infty} a_n$ is convergent.

We should **NOT** infer from the Integral Test that the sum of the series is equal to the value of the integral. In general,

$$\sum_{n=1}^{\infty} a_n \;\ne\; \int_1^\infty f(x)\,dx.$$

Useful Fact

- A **continuous** function is continuous at every point on its **domain**.
  1. Polynomials/Root functions/Trig functions/Exponential functions/Log functions are continuous functions.
  2. If f and g are continuous at a, then $\dfrac{f}{g}$ is continuous at a provided $g \neq 0$.
- If $f'(x) < 0$ on the interval (a,b), then $f(x)$ is **decreasing** on the interval (a,b).

Example:

Use the **Integral Test** to determine the convergence or divergence of the series $\sum_{k=1}^{\infty} \dfrac{k}{k^2+1}$.

***Proof of the p*-series test**

Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

If $p<0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$. If $p=0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.

In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the Divergence Test.

If $p>0$, then the function $f(x) = \frac{1}{x^p}$ is continuous, positive and decreasing on $[1,\infty)$.

We showed on Friday (Sec 7.8 notes): $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p \le 1$.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $0<p\le 1$ by the Integral Test.

***p*-series**

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

convergent if ______________________ and divergent if ______________________ .

Practice/Review:

Determine whether the series $\sum_{k=1}^{\infty} k^{-\frac{3}{4}}$ converges or diverges.

Practice/Review:

Determine whether the series $\sum_{k=4}^{\infty} \frac{1}{(k-1)^{\sqrt{2}}}$ converges or diverges.

Practice/Review:
Which of the following is a convergent p-series?

A.) $\sum_{k=1}^{\infty} \frac{3}{2^k}$ B.) $\sum_{k=1}^{\infty} \frac{3}{\left(\frac{1}{2}\right)^k}$ C.) $\sum_{k=1}^{\infty} \frac{3}{k^2}$ D.) $\sum_{k=1}^{\infty} \frac{3}{k^{\frac{1}{2}}}$

Strategy

Assume $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n > 0$ for all n.

1. Check if it is a **Geometric Series**.
   No! Go to (2).
   Yes! If $r \geq 1$ or $r \leq -1$, then the series diverges. If $-1 < r < 1$, then $S = \frac{a_1}{1-r}$.

2. Check if it is a ***p*-Series**.
   No! Go to (3).
   Yes! If $p \leq 1$, then the series diverges. If $p > 1$, then the series converges.

3. Check if $\lim_{k\to\infty} a_k = 0$. (**L'Hôpital's Rule is used if necessary**)
   Yes! Then the test is inconclusive. Go to (4).
   No! Then the series diverges by the **Divergence Test**.

4. Check if it is a **Telescoping Series**.
   No! Go to (5).
   Yes! Evaluate S_n by cancelling middle terms (**Partial Fraction Decomposition** is used if necessary) and $S = \lim_{n\to\infty} S_n$.

5. Use the following Tests:
   The Limit Comparison Test / The Comparison Test.
   The Ratio Test.
   The Integral Test. (when a_n is "easy to integrate")

Practice: Determine whether the following series converge or diverge.

1. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

2. $\sum_{n=2}^{\infty} \frac{1}{\mathrm{n}(\ln n)^2}$

Extra practice:

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$