Infinite Series

If we add the terms of a sequence $\{a_k\}_{k=1}^{n}$, we get an expression of the form

$$a_1 + a_2 + a_3 + \quad + a_n$$

which is called a **series** and is denoted by

$$\sum_{k=1}^{n} a_k .$$

Does it make sense to talk about the sum of infinitely many terms? We consider the **partial sums**

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$S_n = a_1 + a_2 + a_3 + \quad + a_n = \sum_{k=1}^{n} a_k .$$

If the sequence of partial sums $\{S_n\}$ has a limit L, then the infinite series **converges** to that limit and we write

____ .

If the sequence of partial sums diverges, then the infinite series also **diverges**.

Summary (Notation)

Sequence converges or diverges?

Series converges or diverges?

An important example of an infinite series is the geometric series.

Recall

- Given a **Geometric Sequence** $\{a_k\}_{k=1}^{\infty}$, if the **ratio** is r, then the n-th term can be expressed as $a_n =$ ______________________.
- When ______________________, the sequence converges.

Geometric Series

> **Theorem Partial Sum of Geometric Series**
>
> Given a Geometric Sequence $\{a_k\}_{k=1}^{\infty}$, if the ratio is r, then the sum of the first n term
>
> $$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1} = \underline{\qquad\qquad\qquad\qquad}.$$

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$$

$$r \cdot S_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + a_1 r^n$$

Therefore, $S_n - r \cdot S_n = a_1 - a_1 r^n$

$$S_n = a_1 \cdot \frac{1-r^n}{1-r}.$$

Furthermore,

$$\lim_{n\to\infty} r^n = \begin{cases} \underline{\qquad\qquad} & |r| < 1 \\ \underline{\qquad\qquad} & r = -1 \\ \underline{\qquad\qquad} & |r| > 1 \end{cases} \quad \Rightarrow \quad S = \lim_{n\to\infty} S_n = \begin{cases} \underline{\qquad\qquad} & |r| < 1 \\ \underline{\qquad\qquad} & |r| = 1 \\ \underline{\qquad\qquad} & |r| > 1 \end{cases}$$

Theorem **Geometric Series**

Let r and a be real numbers. If $|r| < 1$, then $\sum_{k=1}^{\infty} ar^{k-1}$ ______________________. If $|r| \geq 1$, then the series diverges.

Caution

- When ______________________, the **Geometric Sequence** converges.
- When ______________________, the **Geometric Series** converges.

Example:

Evaluate the geometric series $\sum_{k=2}^{\infty} \frac{2^k}{3^{k-1}}$ or state that it diverges.

Repeating Decimals

Example:
Write $2.3\overline{17} = 2.3171717\ldots$ as a geometric series and express its value as a fraction (a ratio of two integers).

Exercise: Useful Observations (see list of week 2 notes for solution)

- $0.\overline{38}$ $\qquad$ $1.\overline{38}$
- $0.2\overline{74}$ $\qquad$ $1.2\overline{74}$